The d-bar approach to approximate inverse scattering at fixed energy in three dimensions

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The $\bar\partial$-approach to approximate inverse scattering at fixed energy in three dimensions

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Abstract

We develop the $\bar\partial$-approach to inverse scattering at fixed energy in dimensions $d \geq 3$ of [Beals, Coifman 1985] and [Henkin, Novikov 1987]. As a result we propose a stable method for nonlinear approximate finding a potential v from its scattering amplitude f at fixed energy $E > 0$ in dimension $d = 3$. In particular, in three dimensions we stably reconstruct n-times smooth potential v with sufficient decay at infinity, $n > 3$, from its scattering amplitude f at fixed energy E up to $O(E^{-(n-3-\varepsilon)/2})$ in the uniform norm as $E \to +\infty$ for any fixed arbitrary small $\varepsilon > 0$ (that is with almost the same decay rate of the error for $E \to +\infty$ as in the linearized case near zero potential).

1. Introduction

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{1.1}$$

where

$$v \in W_s^{n,1}(\mathbb{R}^d) \ \text{ for some } \ n \in \mathbb{N}, \ \ n > d-2, \ \ \text{and some } \ s > 0, \tag{1.2}$$

where

$$W_s^{n,1}(\mathbb{R}^d) = \{u : \ \Lambda^s \partial^J u \in L^1(\mathbb{R}^d) \ \text{ for } \ |J| \leq n\}, \tag{1.3}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^{d} J_i, \quad \partial^J u(x) = \frac{\partial^{|J|} u(x)}{\partial x_1^{J_1} \ldots \partial x_d^{J_d}},$$
$$\Lambda^s w(x) = (1 + |x|^2)^{s/2} w(x), \quad x \in \mathbb{R}^d.$$

For equation (1.1) we consider the scattering amplitude $f(k,l)$, where $(k,l) \in \mathcal{M}_E$,

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : \ k^2 = l^2 = E\}, \quad E > 0. \tag{1.4}$$

For definitions of the scattering amplitude see, for example, [F3] and [FM]. Given v, to determine f one can use, in particular, the integral equation (2.5) of Section 2.

In the present work we consider, in particular, the following inverse scattering problem for equation (1.1):

Problem 1. Given f on $\mathcal{M}_E$ at fixed energy $E > 0$, find v on $\mathbb{R}^d$ (at least approximately, but sufficiently stably for numerical implementations).

Note that the Schrödinger equation (1.1) at fixed energy E can be considered also as the acoustic equation at fixed frequency ω, where $E = \omega^2$ (see, for example, Section 5.2

of [HN]). Therefore, Problem 1 is also a basic problem of the monochromatic ultrasonic tomography. Actually, the creation of effective reconstruction methods for inverse scattering in multidimensions (and especially in three dimensions) was formulated as a very important problem many times in the mathematical literature, see, for example, [Gel], [F3], [Gro]. In particular, as it is mentioned in [Gro]: "For example, an efficient inverse scattering algorithm would revolutionize medical diagnostics, making ultrasonic devices at least as efficient as current X-ray analysis". The results of the present work can be considered as a step to this objective.

Suppose, first, that

$$\|v\|_s^{n,1} = \max_{|J|\le n} \|\Lambda^s \partial^J v\|_{L^1(\mathbb{R}^d)} \tag{1.5}$$

is so small for fixed n, s, d of (1.1), (1.2) and some fixed $E_0 > 0$ that the following well-known Born approximation

$$f(k,l) \approx \hat v(k-l), \quad (k,l)\in \mathcal{M}_E, \quad E\ge E_0, \tag{1.6}$$

$$\hat v(p) = \big(\frac{1}{2\pi}\big)^d \int\limits_{\mathbb{R}^d} e^{ipx} v(x)dx, \quad p\in\mathbb{R}^d, \tag{1.7}$$

is completely satisfactory. Then Problem 1 (for fixed $E\ge E_0$) is reduced to finding v from $\hat v$ on $\mathcal{B}_{2\sqrt{E}}$, where

$$\mathcal{B}_r = \{p\in\mathbb{R}^d:\ |p|<r\}, \quad r>0. \tag{1.8}$$

This linearized inverse scattering problem can be solved by the formula

$$v(x) = v_{appr}(x,E) + v_{err}(x,E), \tag{1.9}$$

where

$$v_{appr}(x,E) = \int\limits_{\mathcal{B}_{2\sqrt{E}}} e^{-ipx}\hat v(p)dp, \quad v_{err}(x,E) = \int\limits_{\mathbb{R}^d\setminus\mathcal{B}_{2\sqrt{E}}} e^{-ipx}\hat v(p)dp, \tag{1.10}$$

$x\in\mathbb{R}^d$, $E\ge E_0$. If v satisfies (1.2), $n>d$, and $\|v\|_0^{n,1}\le C$ then

$$|\hat v(p)| \le C_1(n,d)\, C\,(1+|p|)^{-n}, \quad p\in\mathbb{R}^d, \tag{1.11a}$$

$$|v_{err}(x,E)| \le C_2(n,d)\, C\, E^{-(n-d)/2}, \quad x\in\mathbb{R}^d, \quad E\ge E_0, \tag{1.11b}$$

where $C_1(n,d)$, $C_2(n,d)$ are some positive constants and $\|v\|_0^{n,1}$ is defined by (1.5).

If v satisfies (1.2) and, in addition, is compactly supported or exponentially decaying at infinity, then $\hat v$ on $\mathcal{B}_{2\sqrt{E}}$ uniquely determines $\hat v$ on $\mathbb{R}^d\setminus\mathcal{B}_{2\sqrt{E}}$ (at fixed $E>0$) by an analytic continuation and, therefore, in the Born approximation (1.6) f on $\mathcal{M}_E$ (at fixed $E\ge E_0$) uniquely determines v on $\mathbb{R}^d$. However, this determination is not sufficiently stable for direct numerical implementation.

In [No3], [No5] it was shown, in particular, that if v is a bounded, measurable, real function on $\mathbb{R}^d$ and, in addition, is compactly supported or exponentially decaying at infinity, then f on $\mathcal{M}_E$ at fixed $E > 0$ uniquely determines v almost everywhere on $\mathbb{R}^d$ for $d \geq 3$. In [No2], [No3], [No4] similar results were given also for $d = 2$ under the condition that v is sufficiently small in comparison with fixed E. However, these determinations of [No2], [No3], [No4], [No5] are not sufficiently stable for direct numerical implementation (because of the nature of Problem 1 explained already for the linearized case (1.6)). Actually, any precise reconstruction of v, where $v \in C^n(\mathbb{R}^d)$ and $supp\, v \in \mathcal{B}_r$ for fixed $n \in \mathbb{N}$ and $r > 0$, from f on $\mathcal{M}_E$ at fixed $E > 0$ is exponentially unstable (see [Ma]). In the present work we will not discuss such reconstructions in detail.

In [HN] for $d \geq 2$ it was shown, in particular, that if v is measurable, real function on $\mathbb{R}^d$ and $|v(x)| < C(1+|x|)^{-d-\varepsilon}$ for some positive ε and C, then for any fixed E and δ, where $0 < \delta < E$, f on $\cup_{\lambda \in [E-\delta, E+\delta]} \mathcal{M}_\lambda$ uniquely determines $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$. However, unfortunately, this determination of [HN] involves an analytical continuation and, therefore, is not sufficiently stable for direct numerical implementation.

Note also that in [Ch] an efficient numerical algorithm for the reconstruction from multi-frequency scattering data was proposed in two dimensions, but [Ch] gives no rigorous mathematical theorem.

On the other hand in [No6], [No7] we succeeded to give stable approximate solutions of nonlinearized Problem 1 for $d = 2$ and v satisfying (1.2), $n > d = 2$, with the same decay rate of the error terms for $E \to +\infty$ as in the linearized case (1.6), (1.9), (1.10), (1.11b) (or, more precisely, with the error terms decaying as $O(E^{-(n-2)/2})$ in the uniform norm as $E \to +\infty$). Note that in [No6], [No7] we proceed from the fixed-energy inverse scattering reconstruction procedure developed in [No1], [GM], [No2], [No4] for $d = 2$. (In turn, the works [No1], [GM], [No2], [No4] are based, in particular, on the nonlocal Riemann-Hilbert problem approach of [M], the $\bar{\partial}$-approach of [ABF] and some results of [F2], [F3], [GN].) Note also that the reconstruction procedure of [No1], [GM], [No2], [No4] was implemented numerically in [BBMRS], [BMR].

In the present work we succeeded, in particular, to give a stable approximate solution of nonlinearized Problem 1 for $d = 3$ and v satisfying (1.2), $n > d = 3$, with the error term decaying as $O(E^{-(n-3-\varepsilon)/2})$ in the uniform norm as $E \to +\infty$ for any fixed arbitrary small $\varepsilon > 0$ (that is with almost the same decay rate of the error term for $E \to +\infty$ as in the linearized case (1.6), (1.9), (1.10), (1.11b)).

Note that before the works [No6], [No7] in dimension $d = 2$ and the present work in dimension $d = 3$, even for real v of the Schwartz classs on $\mathbb{R}^d$, $d \geq 2$, no result was given, in general, in the literature on finding v on $\mathbb{R}^d$ from f on $\mathcal{M}_E$ with the error decaying more rapidly than $O(E^{-1/2})$ in the uniform norm as $E \to +\infty$ (see related discussion given in [No6]).

The aforementioned result of the present work is a corollary (see Corollary 3 of Section 8) of the method developed in Sections 3,4,5,6,7 (for $d = 3$) for approximate finding $\hat{v}$ on $\mathcal{B}_{2\tau\sqrt{E}}$ from f on $\mathcal{M}_E$ at fixed $E > 0$, where $\tau \in]0, 1[$ is a parameter of our approximate reconstruction. More precisely, in the present work for v satisfying (1.2), $n > 2$, $d = 3$, and $\|v\|_s^{n,1} \leq C$ we give a stable method for approximate finding $\hat{v}$ on $\mathcal{B}_{2\tau\sqrt{E}}$ from f on $\mathcal{M}_E$ (at fixed $E \geq E(s, n, \mu_0, C) \to +\infty$ as $C \to +\infty$) with the error decaying as $O(E^{-(n-\mu_0)/2})$ in the norm $\|\cdot\|_{E,\tau,\mu_0}$ as $E \to +\infty$ for fixed C, μ_0 and τ, where $\mu_0 \geq 2$,

$0 < \tau < \tau(s, n, \mu_0, C) \to 0$ as $C \to +\infty$,

$$\|w\|_{E,\tau,\mu_0} = \sup_{p \in \mathcal{B}_{2\tau\sqrt{E}}} (1 + |p|)^{\mu_0} |w(p)|, \qquad (1.12)$$

see Theorems 1 and 2 of Section 8. Our reconstruction procedure can be summarized as follows (where $d = 3$):

1. From f on $\mathcal{M}_E$ via the Faddeev equation (2.20) we find the Faddeev generalized scattering amplitude $h_\gamma(k, l)$ for $(k, l) \in \mathcal{M}_E$, $\gamma \in \mathbb{S}^{d-1}$, $\gamma k = 0$;

2. From $h_\gamma(k, l)$, $(k, l) \in \mathcal{M}_E$, $\gamma \in \mathbb{S}^{d-1}$, $\gamma k = 0$, $\gamma l = 0$, via formulas (2.8), (5.1), (5.9a), (5.11) and nonlinear integral equation (5.34) derived in Section 5 we find an approximation $\tilde{H}_{E,\tau}$ to the Faddeev generalized "scattering" amplitude H on $\Omega_E^\tau \backslash Re\, \Omega_E^\tau$ for some $\tau \in]0, 1[$, where

$$\begin{aligned} &\Omega_E^\tau = \{k \in \mathbb{C}^d,\ p \in \mathcal{B}_{2\tau\sqrt{E}} :\ k^2 = E,\ p^2 = 2kp\}, \\ &Re\, \Omega_E^\tau = \{k \in \mathbb{R}^d,\ p \in \mathcal{B}_{2\tau\sqrt{E}} :\ k^2 = E,\ p^2 = 2kp\}; \end{aligned} \qquad (1.13)$$

3. From $\tilde{H}_{E,\tau}$ on $\Omega_E^\tau \backslash Re\, \Omega_E^\tau$ via formulas (7.2) we find approximations $\hat{v}_\pm(\cdot, E, \tau)$ to $\hat{v}$ on $\mathcal{B}_{2\tau\sqrt{E}}$.

This reconstruction procedure (with estimates for the difference $\hat{v} - \hat{v}(\cdot, E, \tau)$ on $\mathcal{B}_{2\tau\sqrt{E}}$) is presented in detail in Theorem 1 of Section 8. A stability estimate for this procedure with respect to errors in f is given in Theorem 2 of Section 8. For the case when nonlinear integral equation (5.34) in our reconstruction procedure is approximately solved just by the very first approximation, this reconstruction is, actually, reduced to the approximate reconstruction proposed and implemented numerically in [ABR] proceeding from [HN]. A numerical realization of the strict reconstruction procedure of the present work is now in preparation by the authors of [ABR].

Note also that in Corollary 2 of Section 8 we give a generalization of Theorem 1 to the case of approximate finding $\hat{v}$ on $\mathcal{B}_{2\tau\sqrt{E}}$ from f given only on

$$\mathcal{M}_{E,\tau} = \{(k, l) \in \mathcal{M}_E :\ |k - l| < 2\tau\sqrt{E}\}, \qquad (1.14)$$

where $E > 0$, $0 < \tau < 1$.

In the present work, besides some results of [F3], we proceed from the $\bar{\partial}$-approach to inverse scattering at fixed energy in dimension $d \geq 3$ of [BC] and [HN] and some estimates of [ER1] and [No7].

The present paper is composed as follows. In Section 2 we give some preliminaries concerning the scattering amplitude f and its Faddeev's extension h. Our method for approximate nonlinear inverse scattering at fixed energy in dimension $d = 3$ is developed in Sections 3,4,5,6,7 and is summarized in Section 8. The main technical proofs are given in Sections 9,10,11,12.

In the present work we consider, mainly, the case of the most important dimension $d = 3$. However, only restrictions in time prevent us from generalizing all main results of the present work to the case $d > 3$.

2. Scattering data and some preliminaires

Note that in this paper we always suppose that real degrees of positive real values denote positive real values. Note also that through c_j we shall denote some positive constants (which can be given explicitly).

Consider

$$C^{\alpha,\mu}(\mathbb{R}^d) = \{u \in C(\mathbb{R}^d) : \ \|u\|_{\alpha,\mu} < +\infty\}, \quad \alpha \in]0,1], \ \mu \in \mathbb{R}, \tag{2.1}$$

where

$$\|u\|_{\alpha,\mu} = \|\Lambda^\mu u\|_\alpha, \tag{2.2a}$$

$$\Lambda^\mu u(p) = (1+|p|^2)^{\mu/2} u(p), \quad p \in \mathbb{R}^d, \tag{2.2b}$$

$$\|w\|_\alpha = \sup_{p,\xi \in \mathbb{R}^d, \ |\xi| \le 1} (|w(p)| + |\xi|^{-\alpha}|w(p+\xi) - w(p)|), \tag{2.2c}$$

Consider also $\mathcal{H}_{\alpha,\mu}$ defined as the closure of $C_0^\infty(\mathbb{R}^d)$ (the space of infinitely smooth functions with compact support) in $\|\cdot\|_{\alpha,\mu}$.

Let $\hat{v}$ be defined by (1.7). If v satisfies (1.2), then

$$\hat{v} \in \mathcal{H}_{\alpha,n}(\mathbb{R}^d), \quad \text{where} \ \ \alpha = \min(1,s). \tag{2.3}$$

For equation (1.1), where

$$\hat{v} \in \mathcal{H}_{\alpha,\mu}(\mathbb{R}^d) \ \text{ for some } \ \alpha \in]0,1[\ \text{ and some real } \ \mu > d-2, \tag{2.4}$$

we consider the function $f(k,l)$, where $k,l \in \mathbb{R}^d$, $k^2 = E$, of the classical scattering theory. Given v, to determinate f one can use the following integral equation

$$f(k,l) = \hat{v}(k-l) - \int\limits_{\mathbb{R}^d} \frac{\hat{v}(m-l) f(k,m) dm}{m^2 - k^2 - i0}, \tag{2.5}$$

where $k,l \in \mathbb{R}^d$, $k^2 > 0$, and where at fixed k the function f is sought in $C^{\alpha,\mu}(\mathbb{R}^d)$. In addition, f on $\mathcal{M}_E$ defined by (1.4) is the scattering amplitude for equation (1.1).

Note that

$$\mathcal{M}_E = \mathbb{S}^{d-1}_{\sqrt{E}} \times \mathbb{S}^{d-1}_{\sqrt{E}}, \quad \text{where} \tag{2.6}$$

$$\mathbb{S}^{d-1}_r = \{m \in \mathbb{R}^d : \ |m| = r\}, \quad r > 0. \tag{2.7}$$

For equation (1.1), where $\hat{v}$ satisfies (2.4), we consider also the Faddeev functions

$$\begin{aligned} &h_\gamma(k,l) = H_\gamma(k,k-l), \quad \text{where} \ \ k,l \in \mathbb{R}^d, \ k^2 = E, \ \gamma \in \mathbb{S}^{d-1}, \ \ \text{and} \\ &h(k,l) = H(k,k-l), \quad \text{where} \ \ k,l \in \mathbb{C}^d \backslash \mathbb{R}^d, \ k^2 = E, \ \ Im\, k = Im\, l \\ &(\text{see [F3], [HN], [No7]}) : \end{aligned} \tag{2.8}$$

$$H_\gamma(k,p)=\hat v(p)-\int\limits_{\mathbb{R}^d}\frac{\hat v(p+\xi)H_\gamma(k,-\xi)d\xi}{\xi^2+2(k+i0\gamma)\xi},\quad k\in\mathbb{R}^d\backslash 0,\ \gamma\in\mathbb{S}^{d-1},\ p\in\mathbb{R}^d, \tag{2.9}$$

where at fixed $\gamma\in\mathbb{S}^{d-1}$ and $k\in\mathbb{R}^d\backslash 0$ we consider (2.9) as an equation for $H_\gamma(k,\cdot)\in C^{\alpha,\mu}(\mathbb{R}^d)$,

$$H(k,p)=\hat v(p)-\int\limits_{\mathbb{R}^d}\frac{\hat v(p+\xi)H(k,-\xi)d\xi}{\xi^2+2k\xi},\quad k\in\mathbb{C}^d\backslash\mathbb{R}^d,\ p\in\mathbb{R}^d, \tag{2.10}$$

where at fixed $k\in\mathbb{C}^d\backslash\mathbb{R}^d$ we consider (2.10) as an equation for $H(k,\cdot)\in C^{\alpha,\mu}(\mathbb{R}^d)$. In addition, h on $\{k,l\in\mathbb{C}^d\backslash\mathbb{R}^d:\ Im\,k=Im\,l,\ k^2=l^2=E\}$ or that is the same H on $\Omega_E\backslash Re\,\Omega_E$, where

$$\begin{aligned}&\Omega_E=\{k\in\mathbb{C}^d,\ \ p\in\mathbb{R}^d:\ p^2=2kp,\ k^2=E\},\\ &Re\,\Omega_E=\{k\in\mathbb{R}^d,\ \ p\in\mathbb{R}^d:\ p^2=2kp,\ k^2=E\},\end{aligned} \tag{2.11}$$

can be considered as the scattering amplitude in the complex domain for equation (1.1).

Consider the operator $\tilde A^+(k)$ from (2.5) and the operators $A_\gamma(k)$, $A(k)$ from (2.9), (2.10):

$$(\tilde A^+(k)U)(l)=\int\limits_{\mathbb{R}^d}\frac{\hat v(m-l)U(m)dm}{m^2-k^2-i0},\quad l\in\mathbb{R}^d,\ k\in\mathbb{R}^d\backslash 0, \tag{2.12}$$

$$(A_\gamma(k)U)(p)=\int\limits_{\mathbb{R}^d}\frac{\hat v(p+\xi)U(-\xi)d\xi}{\xi^2+2(k+i0\gamma)\xi},\quad p\in\mathbb{R}^d,\ \gamma\in\mathbb{S}^{d-1},\ k\in\mathbb{R}^d\backslash 0, \tag{2.13}$$

$$(A(k)U)(p)=\int\limits_{\mathbb{R}^d}\frac{\hat v(p+\xi)U(-\xi)d\xi}{\xi^2+2k\xi},\quad p\in\mathbb{R}^d,\quad k\in\mathbb{C}^d\backslash\mathbb{R}^d. \tag{2.14}$$

If $\hat v$ satisfies (2.4), then

$$\begin{aligned}&\|\Lambda_k^\mu\tilde A^+(k)\Lambda_k^{-\mu}u\|_\alpha\le(1/2)|k|^{-\sigma}c_1(\alpha,\mu,\sigma,d)\|\hat v\|_{\alpha,\mu}\|u\|_\alpha,\\ &\text{where}\ \ k\in\mathbb{R}^d,\ k^2\ge 1,\ \ \text{and}\ \ (\Lambda_k u)(l)=(1+|k-l|^2)^{1/2}u(l),\ \ l\in\mathbb{R}^d,\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\|\Lambda^\mu A_\gamma(k)\Lambda^{-\mu}u\|_\alpha\le(1/2)|k|^{-\sigma}c_1(\alpha,\mu,\sigma,d)\|\hat v\|_{\alpha,\mu}\|u\|_\alpha,\\ &\text{for}\ \ \gamma\in\mathbb{S}^{d-1},\ \ k\in\mathbb{R}^d,\ \ k^2\ge 1,\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\|\Lambda^\mu A(k)\Lambda^{-\mu}u\|_\alpha\le|Re\,k|^{-\sigma}c_1(\alpha,\mu,\sigma,d)\|\hat v\|_{\alpha,\mu}\|u\|_\alpha,\\ &\text{for}\ \ k\in\mathbb{C}^d\backslash\mathbb{R}^d,\ \ \mathbb{R}\ni k^2\ge 1,\end{aligned} \tag{2.17}$$

where $u\in C^{\alpha,0}(\mathbb{R}^d)$, $0\le\sigma<min(1,\mu-d+2)$. Estimate (2.15) follows from Theorem 2.1 of [ER1] for $d=3$ and Theorem 1.1 of [ER2] for $d=2$. (Note also that (2.15) is a

development of a related estimate from [F1] for $d = 3$.) Estimates (2.16), (2.17) are given in Proposition 1.1 of [No7].

If v satisfies (2.4) and

$$\|\hat v\|_{\alpha,\mu} \le N < \frac{E^{\sigma/2}}{c_1(\alpha,\mu,\sigma,d)} \ \text{ for some } \ \sigma \in]0, min(1, \mu - d + 2)[\ \text{ and some } \ E \ge 1, \tag{2.18}$$

then the following results are valid:

I. For $k^2 = E$ equations (2.5), (2.9) and (2.10) considered as mentioned above (for fixed k or for fixed k and γ) are uniquely solvable (by the method of successive approximations).

II. The following formulas hold (see [F3], [HN]):

$$\begin{aligned} &h_\gamma(k,l) = h(k + i0\gamma, l + i0\gamma) \ \text{ for } \ k,l \in \mathbb{R}^d, \ k^2 = E, \ \gamma \in \mathbb{S}^{d-1}, \\ &f(k,l) = h_{k/|k|}(k,l) \ \text{ for } \ k,l \in \mathbb{R}^d, \ k^2 = E; \end{aligned} \tag{2.19}$$

$$h_\gamma(k,l) = f(k,l) + \frac{\pi i}{\sqrt{E}} \int\limits_{\mathbb{S}^{d-1}_{\sqrt{E}}} h_\gamma(k,m)\chi((m-k)\gamma) f(m,l) dm, \tag{2.20}$$

where

$$\chi(s) = 0 \ \text{ for } \ s \le 0, \ \chi(s) = 1 \ \text{ for } \ s > 0, \tag{2.21}$$

$k, l \in \mathbb{R}^d$, $k^2 = E$, $\gamma \in \mathbb{S}^{d-1}$, dm is the standard measure on $\mathbb{S}^{d-1}_{\sqrt{E}}$.

III. The following $\bar\partial$- equation holds (see [BC], [HN]):

$$\begin{aligned} &\frac{\partial}{\partial \bar k_j} H(k,p) = -2\pi \int\limits_{\mathbb{R}^d} \xi_j H(k,-\xi) H(k+\xi, p+\xi) \delta(\xi^2 + 2k\xi) d\xi, \\ &j = 1, \dots, d, \quad k \in \mathbb{C}^d \backslash \mathbb{R}^d, \quad k^2 = E, \quad p \in \mathbb{R}^d, \end{aligned} \tag{2.22}$$

where δ is the Dirac function; in addition, for $d \ge 3$

$$\int\limits_{\mathbb{R}^d} u(\xi)\delta(\xi^2 + 2k\xi) d\xi = \int_{\{\xi \in \mathbb{R}^d: \ \xi^2 + 2k\xi = 0\}} \frac{u(\xi)}{|J(k,\xi)|} |d\xi_3 \wedge \dots d\xi_d|, \tag{2.23}$$

where $J(k,\xi) = 4[(\xi_1 + Re\, k_1) Im\, k_2 - (\xi_2 + Re\, k_2) Im\, k_1]$ is the Jacobian of the map $(\xi_1, \dots, \xi_d) \to (\xi^2 + 2Re\, k\xi, 2Im\, k\xi, \xi_3, \dots, \xi_d)$ and u is a test function.

IV. The following estimates are valid:

$$|H(k,p) - \hat v(p)| \le \frac{\eta}{1-\eta} N (1 + p^2)^{-\mu/2}, \tag{2.24}$$

where

$$\eta = |Re\, k|^{-\sigma} c_1(\alpha,\mu,\sigma,d) N, \tag{2.25}$$

$k \in \mathbb{C}^d \backslash \mathbb{R}^d$, $k^2 = E$, $p \in \mathbb{R}^d$ (and where $\eta < 1$ due to (2.18) and the inequality $|Re\, k| \geq E^{1/2}$), and, as a corollary,

$$\hat{v}(p) = H(k,p) + O\big(\frac{1}{(1+p^2)^{\mu/2}|k|^{\sigma}}\big) \ \text{ as } \ |k| = (|Re\, k|^2 + |Im\, k|^2)^{1/2} \to \infty, \tag{2.26}$$

where $k \in \mathbb{C}^d \backslash \mathbb{R}^d$, $k^2 = E$, $p \in \mathbb{R}^d$;

$$|f(k,l)| \leq \frac{N}{1-\eta}(1+|k-l|^2)^{-\mu/2}, \quad k,l \in \mathbb{R}^d, \quad k^2 = E, \tag{2.27a}$$

$$|f(k,l) - \hat{v}(k-l)| \leq \frac{\eta}{1-\eta} N(1+|k-l|^2)^{-\mu/2}, \quad k,l \in \mathbb{R}^d, \ k^2 = E, \tag{2.27b}$$

$$\begin{gathered} |f(k,l) - f(k',l')| \leq \\ c_2(\mu)\frac{N}{1-\eta}(1+|k-l|^2)^{-\mu/2}(|l-l'|^{\alpha} + |k-k'|^{\alpha}), \\ k,k',l,l' \in \mathbb{R}^d, \ k^2 = {k'}^2 = l^2 = {l'}^2 = E, \quad |k-k'| \leq 1, \quad |l-l'| \leq 1, \end{gathered} \tag{2.28a}$$

$$\begin{gathered} |f(k,l) - \hat{v}(k-l) - (f(k',l') - \hat{v}(k'-l'))| \leq \\ c_2(\mu)\frac{\eta}{1-\eta}N(1+|k-l|^2)^{-\mu/2}(|l-l'|^{\alpha} + |k-k'|^{\alpha}), \\ k,k',l,l' \in \mathbb{R}^d, \ k^2 = {k'}^2 = l^2 = {l'}^2 = E, \quad |k-k'| \leq 1, \quad |l-l'| \leq 1, \end{gathered} \tag{2.28b}$$

where

$$c_2(\mu) = c_2'(\mu)c_2''(\mu), \tag{2.28c}$$

$$c_2'(\mu) = \sup_{\substack{p,p' \in \mathbb{R}^d, \\ |p-p'| \leq 1}} \frac{(1+|p|^2)^{\mu/2}}{(1+|p'|^2)^{\mu/2}}, \tag{2.28d}$$

$$c_2''(\mu) = 1 + \sup_{\substack{p,p' \in \mathbb{R}^d, \\ |p-p'| \leq 1}} \frac{|(1+|p|^2)^{\mu/2} - (1+|p'|^2)^{\mu/2}|}{(1+|p'|^2)^{\mu/2}|p-p'|}, \tag{2.28c}$$

$$|H_{\gamma}(k,p) - \hat{v}(p)| \leq \frac{\eta}{1-\eta}N(1+p^2)^{-\mu/2}, \quad k,p \in \mathbb{R}^d, \ k^2 = E, \ \gamma \in \mathbb{S}^{d-1}, \tag{2.29a}$$

$$\begin{gathered} |H_{\gamma}(k,p) - \hat{v}(p) - (H_{\gamma}(k,p') - \hat{v}(p'))| \leq \frac{c_2''(\mu)\eta}{1-\eta}N(1+p^2)^{-\mu/2}|p-p'|^{\alpha}, \\ k,p,p' \in \mathbb{R}^d, \ k^2 = E, \ |p-p'| \leq 1, \ \gamma \in \mathbb{S}^{d-1}, \end{gathered} \tag{2.29b}$$

$$|H(k,p)| \leq \frac{1}{1-\eta}N(1+p^2)^{-\mu/2}, \quad k \in \mathbb{C}^d \backslash \mathbb{R}^d, \ k^2 = E, \ p \in \mathbb{R}^d, \tag{2.30}$$

where

$$\eta = E^{-\sigma/2} c_1(\alpha, \mu, \sigma, d) N \tag{2.31}$$

(and where $\eta < 1$ due to (2.18)).

Estimates (2.27), (2.29)-(2.31) follow from (2.5), (2.9), (2.10), (2.15)-(2.17). Estimate (2.28) follows from (2.5), (2.15) and the symmetry

$$f(k,l) = f(-k,-l), \quad k,l \in \mathbb{R}^d, \quad k^2 = l^2 = E > 0. \tag{2.32}$$

Consider

$$C^\alpha(\mathbb{S}_r^{d-1}) = \{u \in C(\mathbb{S}_r^{d-1}) : \ \|u\|_{C^\alpha(\mathbb{S}_r^{d-1})} < +\infty\}, \ \alpha \in [0,1[, \ r > 0, \tag{2.33}$$

where

$$\begin{aligned}
&\|u\|_{C^\alpha(\mathbb{S}_r^{d-1})} = \|u\|_{C(\mathbb{S}_r^{d-1})} = \sup_{m \in \mathbb{S}_r^{d-1}} |u(m)| \ \text{ for } \ \alpha = 0, \\
&\|u\|_{C^\alpha(\mathbb{S}_r^{d-1})} = \max(\|u\|_{C(\mathbb{S}_r^{d-1})}, \|u\|'_{C^\alpha(\mathbb{S}_r^{d-1})}), \\
&\|u\|'_{C^\alpha(\mathbb{S}_r^{d-1})} = \sup_{\substack{m,m' \in \mathbb{S}_r^{d-1}, \\ |m-m'| \le 1}} |m-m'|^{-\alpha} |u(m) - u(m')| \ \text{ for } \ \alpha \in]0,1[.
\end{aligned} \tag{2.34}$$

Consider

$$C^\alpha(\mathcal{M}_E) = \{u \in C(\mathcal{M}_E) : \ \|u\|_{C^\alpha(\mathcal{M}_E),0} < +\infty\}, \quad \alpha \in [0,1[, \ E > 0, \tag{2.35}$$

where

$$\begin{aligned}
&\|u\|_{C^\alpha(\mathcal{M}_E),\mu} = \|u\|_{C(\mathcal{M}_E),\mu} = \\
&\sup_{(k,l) \in \mathcal{M}_E} (1 + |k-l|^2)^{\mu/2} |u(k,l)| \ \text{ for } \ \alpha = 0, \ \mu \ge 0, \\
&\|u\|_{C^\alpha(\mathcal{M}_E),\mu} = \max(\|u\|_{C(\mathcal{M}_E),\mu}, \|u\|'_{C^\alpha(\mathcal{M}_E),\mu}), \\
&\|u\|'_{C^\alpha(\mathcal{M}_E),\mu} = \sup_{\substack{(k,l),(k',l') \in \mathcal{M}_E, \\ |k-k'| \le 1, |l-l'| \le 1}} (1 + |k-l|^2)^{\mu/2} (|k-k'|^\alpha + |l-l'|^\alpha)^{-1} \times \\
&|u(k,l) - u(k',l')| \ \text{ for } \ \alpha \in]0,1[, \ \mu \ge 0.
\end{aligned} \tag{2.36}$$

If assumptions (2.4), (2.18) are fulfilled, then, as a corollary of (2.27), (2.28),

$$\begin{aligned}
&f \in C^\alpha(\mathcal{M}_E), \quad \|f - \hat{v}\|_{C^\alpha(\mathcal{M}_E),\mu} \le c_2(\mu) \frac{\eta}{1-\eta} N, \\
&\|f\|_{C^\alpha(\mathcal{M}_E),\mu} \le \frac{c_2(\mu)}{1-\eta} N,
\end{aligned} \tag{2.37}$$

where η is given by (2.31), $\hat{v} = \hat{v}(k-l)$, $(k,l) \in \mathcal{M}_E$.

Consider (2.20) for $k^2 = l^2 = E$ as a family of equations parametrized by γ and k for finding $h_\gamma(k,\cdot)$ on $\mathbb{S}^{d-1}_{\sqrt{E}}$ from the scattering amplitude f on $\mathcal{M}_E$. Consider the operator $B_\gamma(k)$ from (2.20):

$$\begin{aligned}(B_\gamma(k)U)(l) &= \frac{\pi i}{\sqrt{E}} \int\limits_{\mathbb{S}^{d-1}_{\sqrt{E}}} U(m)\chi((m-k)\gamma)f(m,l)dm,\\ l\in\mathbb{S}^{d-1},\ \ &\gamma\in\mathbb{S}^{d-1},\ \ k\in\mathbb{R}^d,\ \ k^2=E>0.\end{aligned}\tag{2.38}$$

As a variation of a related result of [F3], we have that if $\hat v$ satisfies (2.4) and if at fixed $\gamma\in\mathbb{S}^{d-1}$ and $k\in\mathbb{R}^d$, $k^2=E>0$, equations (2.5), (2.9) (considered as mentioned above) are uniquely solvable, then (2.20) is uniquely solvable in $C^\beta(\mathbb{S}^{d-1}_{\sqrt{E}})$ for any fixed $f\in[0,\alpha]$. Besides, the following estimate holds:

$$\begin{aligned}&\|\Lambda_k^\mu B_\gamma(k)\Lambda_k^{-\mu}u\|_{C^\beta(\mathbb{S}^{d-1}_{\sqrt{E}})} \le c_3(\beta,\mu,\sigma,d)E^{-\sigma/2}\|f\|_{C^\beta(\mathcal{M}_E),\mu}\times\\ &\|u\|_{C(\mathbb{S}^{d-1}_{\sqrt{E}})}\ \text{ for }\ f\in C^\beta(\mathcal{M}_E),\ u\in C(\mathbb{S}^{d-1}_{\sqrt{E}}),\end{aligned}\tag{2.39}$$

where $\beta\in[0,1[$, $\mu>d-2$, $0\le\sigma<\min(1,\mu-d+2)$, $\gamma\in\mathbb{S}^{d-1}$, $k\in\mathbb{R}^d$, $k^2=E\ge1$, and

$$(\Lambda_k u)(l) = (1+|k-l|^2)^{1/2}u(l),\ \ l\in\mathbb{S}^{d-1}_{\sqrt{E}}.\tag{2.40}$$

In (2.39) we do not assume that f is related with $\hat v$. Actually, (2.39) is simpler than (2.15), (2.16), because (2.39) contains no singular integrals.

3. Coordinates on Ω_E^τ for $E>0$, $\tau\in]0,1]$ and $d=3$

Consider Ω_E^τ and $Re\,\Omega_E^\tau$ defined by (1.13). Note that $\Omega_E^{\tau_1}\subset\Omega_E^{\tau_2}$, $Re\,\Omega_E^{\tau_1}\subset Re\,\Omega_E^{\tau_2}$, $\tau_1<\tau_2$, and $\Omega_E^{+\infty}=\Omega_E$, $Re\,\Omega_E^{+\infty}=Re\,\Omega_E$, where Ω_E and $Re\,\Omega_E$ are defined by (2.11).

For our considerations for $d=3$ we introduce some convenient coordinates on Ω_E^τ, $E>0$, $0<\tau\le1$. Let

$$\begin{aligned}\Omega_{E,\nu}^\tau &= \{k\in\mathbb{C}^3,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu:\ k^2=E,\ p^2=2kp\},\\ Re\,\Omega_{E,\nu}^\tau &= \{k\in\mathbb{R}^3,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu:\ k^2=E,\ p^2=2kp\},\end{aligned}\tag{3.1}$$

where $E>0$, $0<\tau\le1$,

$$\mathcal{B}_r=\{p\in\mathbb{R}^3:\ |p|<r\},\ r>0,\tag{3.2}$$

$$\mathcal{L}_\nu=\{x\in\mathbb{R}^3:\ t\nu,\ t\in\mathbb{R}\},\ \ \nu\in\mathbb{S}^2.\tag{3.3}$$

Note that $\Omega_{E,\nu}^\tau$ is an open and dense subset of Ω_E^τ for $d=3$, $E>0$, $\tau\in]0,1]$.

For $p\in\mathbb{R}^3\backslash\mathcal{L}_\nu$ consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned}&\theta(p),\ \omega(p)\ \text{ smoothly depend on }\ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\\ &\text{take their values in }\ \mathbb{S}^2\ \text{ and}\\ &\theta(p)p=0,\ \omega(p)p=0,\ \theta(p)\omega(p)=0.\end{aligned}\tag{3.4}$$

Note that (3.4) implies that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \quad \text{for} \quad p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu \tag{3.5a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \quad \text{for} \quad p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \tag{3.5b}$$

where $\times$ denotes vector product.

To satisfy (3.4), (3.5a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \ \omega(p) = \frac{p \times \theta(p)}{|p|}, \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu. \tag{3.6}$$

Lemma 1. *Let $E > 0$, $\nu \in \mathbb{S}^2$. Let θ, ω satisfy* (3.4). *Then the following formulas give a diffeomorphism between $\Omega^\tau_{E,\nu}$ and $(\mathbb{C}\backslash 0) \times (\mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu)$ for $\tau \in]0, 1]$:*

$$(k, p) \to (\lambda, p), \quad \textit{where} \quad \lambda = \lambda(k, p) = \frac{k(\theta(p) + i\omega(p))}{(E - p^2/4)^{1/2}}, \tag{3.7}$$

$$\begin{gathered}(\lambda, p) \to (k, p), \quad \textit{where} \quad k = k(\lambda, p, E) = \kappa_1(\lambda, p, E)\theta(p) + \kappa_2(\lambda, p, E)\omega(p) + p/2, \\ \kappa_1(\lambda, p, E) = (\lambda + 1/\lambda)\frac{(E - p^2/4)^{1/2}}{2}, \ \kappa_2(\lambda, p, E) = (1/\lambda - \lambda)\frac{i(E - p^2/4)^{1/2}}{2},\end{gathered} \tag{3.8}$$

where $(k, p) \in \Omega^\tau_{E,\nu}$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu)$. In addition, formulas (3.7), (3.8) *give also diffeomorphisms between $Re\,\Omega^\tau_{E,\nu}$ and $\mathcal{T} \times (\mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu)$ and between $\Omega^\tau_{E,\nu} \backslash Re\,\Omega^\tau_{E,\nu}$ and $(\mathbb{C}\backslash(0 \cup \mathcal{T})) \times (\mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu)$ for $\tau \in]0, 1]$, where*

$$\mathcal{T} = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}. \tag{3.9}$$

Actually, Lemma 1 follows from properties (3.4) and the result that formulas (3.7), (3.8) for $\lambda(k)$ and $k(\lambda)$ at fixed $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$ give a diffeormorphism between $\{k \in \mathbb{C}^3 : k^2 = E, \ p^2 = 2kp\}$ and $\mathbb{C}\backslash 0$. The latter result follows from the fact (see [No4]) that the following formulas

$$\lambda = \frac{(k_1 + ik_2)}{E^{1/2}}, \quad k_1 = (\lambda + 1/\lambda)\frac{E^{1/2}}{2}, \quad k_2 = (1/\lambda - \lambda)\frac{iE^{1/2}}{2}$$

give a diffeomorphism between $\{k \in \mathbb{C}^2 : \ k^2 = E\}$, $E > 0$, and $\mathbb{C}\backslash 0$.

Note that for k and λ of (3.7), (3.8) the following formulas hold:

$$\begin{aligned}|Im\,k| &= \frac{(E - p^2/4)^{1/2}}{2}\big||\lambda| - 1/|\lambda|\big|, \\ |Re\,k| &= \left(\frac{E - p^2/4}{4}(|\lambda| + 1/|\lambda|)^2 + p^2/4\right)^{1/2},\end{aligned} \tag{3.10}$$

where $(k,p)\in\Omega^\tau_{E,\nu}$, $(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu)$, $E>0$, $\nu\in\mathbb{S}^2$, $\tau\in]0,1]$.

We consider λ, p of Lemma 1 as coordinates on $\Omega^\tau_{E,\nu}$ and on Ω^τ_E for $E>0$, $\tau\in]0,1]$, $d=3$.

4. $\bar\partial$-equation for H on $\Omega^\tau_E\backslash Re\,\Omega^\tau_E$ for $E>0$, $\tau\in]0,1[$ and $d=3$

Lemma 2. *Let $\nu\in\mathbb{S}^2$ and let θ, ω satisfy* (3.4), (3.5a). *Let assumptions* (2.4), (2.18) *be fulfilled. Then*

$$\begin{aligned}
&\frac{\partial}{\partial\bar\lambda}H(k(\lambda,p,E),p)=\\
&-\frac{\pi}{4}\int\limits_{-\pi}^{\pi}\left((E-p^2/4)^{1/2}\frac{sgn\,(|\lambda|^2-1)(|\lambda|^2+1)}{\bar\lambda|\lambda|}(\cos\varphi-1)-|p|\frac{1}{\bar\lambda}\sin\varphi\right)\times\\
&H(k(\lambda,p,E),-\xi(\lambda,p,E,\varphi))H(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))d\varphi
\end{aligned}\tag{4.1}$$

for $\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau\in]0,1]$, where λ and p are coordinates of Lemma 1, $k(\lambda,p,E)$ is defined in (3.8) (*and also depends on ν, θ, ω*),

$$\xi(\lambda,p,E,\varphi)=Re\,k(\lambda,p,E)(\cos\varphi-1)+k^\perp(\lambda,p,E)\sin\varphi,\tag{4.2}$$

$$k^\perp(\lambda,p,E)=\frac{Im\,k(\lambda,p,E)\times Re\,k(\lambda,p,E)}{|Im\,k(\lambda,p,E)|}.\tag{4.3}$$

Proof of Lemma 2 is given in Section 9. In this proof we deduce (4.1) from (2.22).

Note that on the left-hand side of (4.1)

$$(k(\lambda,p,E),p)\in\Omega^\tau_{E,\nu}\backslash Re\,\Omega^\tau_{E,\nu},$$

whereas on the right-hand side of (4.1)

$$\begin{aligned}
&(k(\lambda,p,E),-\xi(\lambda,p,E,\varphi))\in\Omega_E\backslash Re\,\Omega_E,\\
&(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))\in\Omega_E\backslash Re\,\Omega_E,
\end{aligned}$$

but not necessarily

$$\begin{aligned}
&(k(\lambda,p,E),-\xi(\lambda,p,E,\varphi))\in\Omega^\tau_E\backslash Re\,\Omega^\tau_E,\\
&(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))\in\Omega^\tau_E\backslash Re\,\Omega^\tau_E
\end{aligned}$$

for $\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau\in]0,1]$, $\varphi\in]-\pi,\pi[$.

Therefore, consider $\chi_r H$, where χ_r denotes the multiplication operator by the function $\chi_r(p)$, where

$$\chi_r(p)=1\ \text{ for }\ |p|<r,\quad \chi_r(p)=0\ \text{ for }\ |p|\ge r,\quad\text{where}\ \ p\in\mathbb{R}^3,\ r>0.\tag{4.4}$$

Note that

$$\begin{aligned}
&\chi_{2\tau\sqrt{E}}H(k,p)=H(k,p)\ \text{ for }\ (k,p)\in\Omega^\tau_E\backslash Re\,\Omega^\tau_E,\\
&\chi_{2\tau\sqrt{E}}H(k,p)=0\ \text{ for }\ (k,p)\in(\Omega_E\backslash Re\,\Omega_E)\backslash\Omega^\tau_E,
\end{aligned}\tag{4.5}$$

where $E > 0$, $\tau \in]0,1]$.

Further, (under the assumptions of Lemma 1) consider

$$
\begin{aligned}
&\{U_1, U_2\}(\lambda, p, E) = \\
&-\frac{\pi}{4}\int\limits_{-\pi}^{\pi}\left((E - p^2/4)^{1/2}\frac{sgn\,(|\lambda|^2 - 1)(|\lambda|^2 + 1)}{\bar\lambda|\lambda|}(\cos\varphi - 1) - |p|\frac{1}{\bar\lambda}\sin\varphi\right)\times \qquad (4.6)\\
&U_1(k(\lambda, p, E), -\xi(\lambda, p, E, \varphi))U_2(k(\lambda, p, E) + \xi(\lambda, p, E, \varphi), p + \xi(\lambda, p, E, \varphi))d\varphi,
\end{aligned}
$$

where U_1 and U_2 are test functions on $\Omega_E \backslash Re\,\Omega_E$, $k(\lambda, p, E)$ and $\xi(\lambda, p, E, \varphi)$ are defined by (3.8), (4.2), $\lambda \in \mathbb{C}\backslash(\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1]$. Now we can write (4.1) as

$$
\frac{\partial}{\partial\bar\lambda}\chi_{2\tau\sqrt{E}}H(k(\lambda, p, E), p) = \{H, H\}(\lambda, p, E) = \{\chi_{2\tau\sqrt{E}}H, \chi_{2\tau\sqrt{E}}H\}(\lambda, p, E) + R_{E,\tau}(\lambda, p), \qquad (4.7)
$$

$$
\begin{aligned}
&R_{E,\tau}(\lambda, p) = \\
&\{(1 - \chi_{2\tau\sqrt{E}})H, \chi_{2\tau\sqrt{E}}H\}(\lambda, p, E) + \{\chi_{2\tau\sqrt{E}}H, (1 - \chi_{2\tau\sqrt{E}})H\}(\lambda, p, E) + \qquad (4.8)\\
&\{(1 - \chi_{2\tau\sqrt{E}})H, (1 - \chi_{2\tau\sqrt{E}})H\}(\lambda, p, E),
\end{aligned}
$$

where $\lambda \in \mathbb{C}\backslash(\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1]$. Thus, (4.7) is a $\bar\partial$-equation for H on $\Omega_E^\tau \backslash Re\,\Omega_E^\tau$ up to the remainder $R_{E,\tau}$. Let us estimate $R_{E,\tau}$ under assumptions (2.4), (2.18) for $d = 3$, $\mu \geq 2$ (see estimate (4.14) given below).

Let

$$
|||U|||_{E,\mu} = \sup_{(k,p)\in\Omega_E\backslash Re\,\Omega_E}(1 + |p|)^\mu\,|U(k, p)| \qquad (4.9)
$$

for $U \in L^\infty(\Omega_E \backslash Re\,\Omega_E)$, $E > 0$, $\mu \geq 0$.

Lemma 3. *Let the assumptions of Lemma 1 be fulfilled. Let* $U_1, U_2 \in L^\infty(\Omega_E\backslash Re\,\Omega_E)$, $|||U_1|||_{E,\mu} < \infty$, $|||U_2|||_{E,\mu} < \infty$ *for some* $\mu \geq 2$. *Then*:

$$
\{U_1, U_2\}(\cdot, \cdot, E) \in L^\infty((\mathbb{C}\backslash(\mathcal{T} \cup 0)) \times (\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)), \quad \tau \in]0, 1[, \qquad (4.10)
$$

and

$$
|\{U_1, U_2\}(\lambda, p, E)| \leq \frac{c_4(\mu, \tau, E)|||U_1|||_{E,\mu}|||U_2|||_{E,\mu}}{(1 + |p|)^\mu(1 + |\lambda|^2)}, \qquad (4.11)
$$

$\lambda \in \mathbb{C}\backslash(\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1[$, *where* c_4 *is some positive constant such that*

$$
c_4(\mu, \tau, E) \leq \frac{c_4'(\mu)}{\sqrt{z}}\big(1 + \frac{1}{\sqrt{z}}\big) + \frac{c_4''(\mu)}{\sqrt{E}\min(1, 2\sqrt{z})}, \quad z = \frac{1 - \tau^2}{4\tau^2}, \qquad (4.12)
$$

for some positive constants c_4' *and* c_4''.

Proof of Lemma 3 is given in Section 10.

Remark 1. Using the proof (given in Section 10) of (4.10) one can see also that the variations of U_1, U_2 on the sets of zero measure in $\Omega_E\backslash Re\,\Omega_E$ imply variations of $\{U_1, U_2\}(\cdot, \cdot, E)$ on sets of zero measure, only, in $(\mathbb{C}\backslash(\mathcal{T} \cup 0)) \times (\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)$.

Under assumptions (2.4), (2.18) for $d=3$, due to (2.30), (2.31), (4.4) we have that

$$|||H|||_{E,\mu} \le 2^{\mu/2}(1-\eta)^{-1}N,\ \ |||\chi_{2\tau\sqrt{E}}H|||_{E,\mu} \le 2^{\mu/2}(1-\eta)^{-1}N, \qquad (4.13a)$$

$$|||(1-\chi_{2\tau\sqrt{E}})H|||_{E,\mu_0} \le 2^{\mu/2}(1-\eta)^{-1}(1+2\tau\sqrt{E})^{\mu_0-\mu}N, \qquad (4.13b)$$

where η is given by (2.31), $\tau \in]0,1]$, $0 \le \mu_0 \le \mu$.

Under assumptions (2.4), (2.18), $d=3$, $\mu \ge 2$, using estimates (4.13a), (4.13b) and Lemma 3 we obtain the following estimate for $R_{E,\tau}$:

$$|R_{E,\tau}(\lambda,p)| \le \frac{c_4(\mu_0,\tau,E)2^{\mu}N^2}{(1-\eta)^2(1+2\tau\sqrt{E})^{\mu-\mu_0}(1+|p|)^{\mu_0}(1+|\lambda|^2)}\left(2+\frac{1}{(1+2\tau\sqrt{E})^{\mu-\mu_0}}\right), \qquad (4.14)$$

where $\lambda \in \mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_{\nu}$, $\tau \in]0,1[$, $2 \le \mu_0 \le \mu$.

Estimate (4.14) for sufficiently great $\mu-\mu_0$ and formulas (4.12), (2.31) show, in particular, that $R_{E,\tau}$ rapidly vanishes when $\tau\sqrt{E}$ increases for $0<\tau\le\tau_1$ and fixed $\tau_1<1$.

5. Approximate finding H on $\Omega_E^{\tau}\backslash Re\,\Omega_E^{\tau}$ from $H_{\pm}$ on $Re\,\Omega_E^{\tau}$ for $E>0$, $\tau\in]0,1[$ and $d=3$

Our next purpose is to obtain an integral equation for approximate finding H on $\Omega_E^{\tau}\backslash Re\,\Omega_E^{\tau}$ from $H_{\pm}$ on $Re\,\Omega_E^{\tau}$ for $E>0$, $\tau\in]0,1[$, $d=3$, where

$$H_{\pm}(k,p) = H_{\gamma^{\pm}(k,p)}(k,p), \quad \gamma^{\pm}(k,p) = \pm\frac{p\times(k-p/2)}{|p|\,|k-p/2|}, \qquad (5.1)$$

$(k,p)\in Re\,\Omega_E^1$, $|p|\ne 0$, $E>0$, $d=3$ (where H_{γ} is defined by means of (2.9)). Note that

$$(k-p/2)p = 0, \quad |k-p/2| = (E-p^2/4)^{1/2} \ \text{ for } \ (k,p)\in Re\,\Omega_E^1, \quad E>0. \qquad (5.2)$$

We will give the aforementioned integral equation in the coordinates λ, p of Lemma 1 under assumptions (3.5a). Note that in the coordinates λ, p of Lemma 1 under assumption (3.5a) the following formulas hold:

$$H_{\pm}(k(\lambda,p,E),p) = H(k(\lambda(1\mp 0),p,E),p), \qquad (5.3)$$

$$\gamma^{\pm}(k(\lambda,p,E),p) = \pm\left(\frac{-i}{2}\left(\frac{1}{\lambda}-\lambda\right)\theta(p)+\frac{1}{2}\left(\lambda+\frac{1}{\lambda}\right)\omega(p)\right), \qquad (5.4)$$

where $\lambda\in\mathcal{T}$, $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_{\nu}$.

Consider

$$\mathcal{D}_+ = \{\lambda\in\mathbb{C}:\ |\lambda|<1\}, \quad \mathcal{D}_- = \{\lambda\in\mathbb{C}:\ |\lambda|>1\}. \qquad (5.5)$$

We will use the following formulas:

$$u_+(\lambda) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}} u_+(\zeta)\frac{d\zeta}{\zeta-\lambda} - \frac{1}{\pi}\iint_{\mathcal{D}_+}\frac{\partial u_+(\zeta)}{\partial\bar\zeta}\frac{d\,Re\,\zeta\,d\,Im\,\zeta}{\zeta-\lambda}, \quad \lambda\in\mathcal{D}_+\backslash 0, \qquad (5.6a)$$

$$u_-(\lambda) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}} u_-(\zeta)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)} - \frac{1}{\pi}\iint_{\mathcal{D}_-}\frac{\partial u_-(\zeta)}{\partial\bar\zeta}\frac{\lambda\,d\,Re\,\zeta\,d\,Im\,\zeta}{\zeta(\zeta-\lambda)}, \quad \lambda\in\mathcal{D}, \qquad (5.6b)$$

where $u_+(\lambda)$ is continuous and bounded for $0 < |\lambda| \le 1$, $\partial u_+(\lambda)/\partial\bar\lambda$ is bounded for $0 < |\lambda| < 1$, $u_-(\lambda)$ is continuous and bounded for $|\lambda| \ge 1$, $\partial u_-(\lambda)/\partial\bar\lambda$ is bounded for $|\lambda| > 1$ and $\partial u_-(\lambda)/\partial\bar\lambda = O(|\lambda|^{-2})$ as $|\lambda| \to \infty$ (and where the integrals along the circle $\mathcal{T}$ are taken in the counter-clockwise direction). Note that the aforementioned assumptions on $u_\pm$ in (5.6) can be somewhat weakened. Formulas (5.6) follow from the well-known Cauchy-Green formula

$$u(\lambda) = \frac{1}{2\pi i}\int\limits_{\partial\mathcal{D}} u(\zeta)\frac{d\zeta}{\zeta-\lambda} - \frac{1}{\pi}\iint_{\mathcal{D}}\frac{\partial u(\zeta)}{\partial\bar\zeta}\frac{d\,Re\,\zeta\,d\,Im\,\zeta}{\zeta-\lambda}, \quad \lambda\in\mathcal{D}, \tag{5.7}$$

where $\mathcal{D}$ is a bounded open domain in $\mathbb{C}$ with sufficiently regular boundary $\partial\mathcal{D}$ and u is a sufficiently regular function in $\bar{\mathcal{D}} = \mathcal{D}\cup\partial\mathcal{D}$. Assuming that λ, p are the coordinates of Lemma 1 under assumption (3.5a), consider

$$H(\lambda,p,E) = H(k(\lambda,p,E),p), \quad \lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0), \tag{5.8}$$

$$H_\pm(\lambda,p,E) = H_\pm(k(\lambda,p,E),p), \quad \lambda\in\mathcal{T}, \tag{5.9a}$$

$$H_+(\lambda,p,E) = H(k(\lambda,p,E),p), \quad \lambda\in\mathcal{D}_+\backslash 0, \tag{5.9b}$$

$$H_-(\lambda,p,E) = H(k(\lambda,p,E),p), \quad \lambda\in\mathcal{D}_-, \tag{5.9c}$$

where $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$ and $H_\pm(k(\lambda,p,E),p)$ in (5.9a) are defined using (5.1) or (5.3). One can show that, under assumptions (2.4), (2.18), $d=3$, and for $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$, $H_+(\lambda,p,E)$ is continuous for $0<|\lambda|\le 1$ and $H_-(\lambda,p,E)$ is continuous for $|\lambda|\ge 1$. Further, using (2.30), (4.7), (4.8), (4.13) and Lemma 3 one can see that, under assumptions (2.4), (2.18), where $d=3$, $\mu\ge 2$, and for $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$, $H_\pm(\lambda,p,E)$ satisfy the assumptions for $u_\pm(\lambda)$ mentioned in (5.6) and (therefore) formulas (5.6) hold for $u_\pm(\lambda) = H_\pm(\lambda,p,E)$.

Proposition 1. *Let assumptions* (2.4), (2.18), *where* $d=3$ $\mu\ge 2$, *be fulfilled. Let* λ, p *be the coordinates of Lemma 1 under assumption* (3.5a). *Let* $H(\lambda,p,E)$ *be defined by* (5.8). *Then* $H_{E,\tau} = H(\lambda,p,E)$ *as a function of* $\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0)$ *and* $p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, *where* $\tau\in]0,1[$, *satisfies the following nonlinear integral equation*

$$H_{E,\tau} = H^0_{E,\tau} + M_{E,\tau}(H_{E,\tau}) + Q_{E,\tau}, \quad \tau\in]0,1[, \tag{5.10}$$

where:

$$H^0_{E,\tau}(\lambda,p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}} H_+(\zeta,p,E)\frac{d\zeta}{\zeta-\lambda}, \ \lambda\in\mathcal{D}_+\backslash 0, \ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.11a}$$

$$H^0_{E,\tau}(\lambda,p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}} H_-(\zeta,p,E)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)}, \ \lambda\in\mathcal{D}_-, \ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.11b}$$

where $H_\pm(\lambda,p,E)$ *are defined by* (5.9a);

$$\begin{aligned} &M_{E,\tau}(U)(\lambda,p) = M^+_{E,\tau}(U)(\lambda,p) = \\ &-\frac{1}{\pi}\iint_{\mathcal{D}_+}(U,U)_{E,\tau}(\zeta,p)\frac{d\,Re\,\zeta\,d\,Im\,\zeta}{\zeta-\lambda}, \quad \lambda\in\mathcal{D}_+\backslash 0, \ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \end{aligned} \tag{5.12a}$$

$$M_{E,\tau}(U)(\lambda,p) = M^-_{E,\tau}(U)(\lambda,p) =$$
$$-\frac{1}{\pi}\iint_{\mathcal{D}_-}(U,U)_{E,\tau}(\zeta,p)\frac{\lambda d\,Re\,\zeta\, d\,Im\,\zeta}{\zeta(\zeta-\lambda)}, \quad \lambda\in\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.12b}$$

$$(U_1,U_2)_{E,\tau}(\lambda,p) =$$
$$-\frac{\pi}{4}\int\limits_{-\pi}^{\pi}\left((E-p^2/4)^{1/2}\frac{sgn\,(|\lambda|^2-1)(|\lambda|^2+1)}{\bar\lambda|\lambda|}(\cos\varphi-1)-|p|\frac{1}{\bar\lambda}\sin\varphi\right)\times$$
$$U_1(z_1(\lambda,p,E,\varphi),-\xi(\lambda,p,E,\varphi))U_2(z_2(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))\times$$
$$\chi_{2\tau\sqrt{E}}(\xi(\lambda,p,E,\varphi))\chi_{2\tau\sqrt{E}}(p+\xi(\lambda,p,E,\varphi))d\varphi,\quad \lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0),\quad p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.13}$$

where U, U_1, U_2 *are test functions on* $(\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)$,

$$z_1(\lambda,p,E,\varphi) = \frac{k(\lambda,p,E)(\theta(-\xi(\lambda,p,E,\varphi))+i\omega(-\xi(\lambda,p,E,\varphi)))}{(E-|\xi(\lambda,p,E,\varphi)|^2/4)^{1/2}},$$
$$z_2(\lambda,p,E,\varphi) = \frac{(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi))(\theta(p+\xi(\lambda,p,E,\varphi))+i\omega(p+\xi(\lambda,p,E,\varphi)))}{(E-|p+\xi(\lambda,p,E,\varphi)|^2/4)^{1/2}} \tag{5.14}$$

for $\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$, $\varphi\in[-\pi,\pi]$, *and where* $k(\lambda,p,E)$, $\xi(\lambda,p,E,\varphi)$ *are defined by* (3.8), (4.2), θ, ω *are the vector functions of* (3.4), (3.5a);

$$Q_{E,\tau}(\lambda,p) = -\frac{1}{\pi}\iint_{\mathcal{D}_+}R_{E,\tau}(\zeta,p)\frac{d\,Re\,\zeta\, d\,Im\,\zeta}{\zeta-\lambda}, \quad \lambda\in\mathcal{D}_+\backslash 0,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.15a}$$
$$Q_{E,\tau}(\lambda,p) = -\frac{1}{\pi}\iint_{\mathcal{D}_-}R_{E,\tau}(\zeta,p)\frac{\lambda d\,Re\,\zeta\, d\,Im\,\zeta}{\zeta(\zeta-\lambda)}, \quad \lambda\in\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{5.15b}$$

where $R_{E,\tau}$ *is defined by* (4.8).

Proposition 1 follows from formulas (5.6) for $u_\pm(\lambda) = H_\pm(\lambda,p,E)$ (defined by (5.9)), the $\bar\partial$- equation (4.7), formula (4.6) and Lemma 1. We consider (5.10) as an integral equation for finding $H_{E,\tau}$ from $H^0_{E,\tau}$ with unknown remainder $Q_{E,\tau}$. Thus, actually, we consider (5.10) as an approximate equation for finding $H_{E,\tau}$ from $H^0_{E,\tau}$.

Let
$$|||U|||_{E,\tau,\mu} = \sup_{\substack{\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0),\\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu}}(1+|p|)^\mu|U(\lambda,p)| \tag{5.16}$$

for $U\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, where $E>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$, $\mu>0$.

Lemma 4. *Let* $E>0$, $\nu\in\mathbb{S}^2$, $\tau\in]0,1[$, $\mu\geq 2$. *Let* $M_{E,\tau}$ *be defined by* (5.12) (*where* λ, p *are the coordinates of Lemma 1 under assumption* (3.5a)). *Let* $U_1,U_2\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, $|||U_1|||_{E,\tau,\mu}<+\infty$, $|||U_2|||_{E,\tau,\mu}<+\infty$. *Then*

$$M_{E,\tau}(U_j)\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)),\ j=1,2, \tag{5.17}$$
$$|||M_{E,\tau}(U_j)|||_{E,\tau,\mu}\leq c_5c_4(\mu,\tau,E)(|||U_j|||_{E,\tau,\mu})^2,\ j=1,2, \tag{5.18}$$

$$\begin{aligned} &|||M_{E,\tau}(U_1) - M_{E,\tau}(U_2)|||_{E,\tau,\mu} \le \\ &\le c_5 c_4(\mu,\tau,E)(|||U_1|||_{E,\tau,\mu} + |||U_2|||_{E,\tau,\mu})|||U_1 - U_2|||_{E,\tau,\mu}, \end{aligned} \tag{5.19}$$

where $c_4(\mu,\tau,E)$ *is the constant of* (4.11) *and*

$$c_5 = \sup_{\lambda\in\mathcal{D}_+} \frac{1}{\pi}\iint_{\mathcal{D}_+} \frac{d\,Re\,\zeta\,d\,Im\,\zeta}{(1+|\zeta|^2)|\zeta-\lambda|} = \sup_{\lambda\in\mathcal{D}_-} \frac{1}{\pi}\iint_{\mathcal{D}_-} \frac{|\lambda| d\,Re\,\zeta\,d\,Im\,\zeta}{|\zeta|(1+|\zeta|^2)|\zeta-\lambda|}. \tag{5.20}$$

Formulas (5.17), (5.18) follow from formulas (5.12) (5.13), Lemma 1, formula (4.6) and Lemma 3. To obtain (5.19) we use also the formula

$$(U_1,U_1)_{E,\tau} - (U_2,U_2)_{E,\tau} = (U_1-U_2,U_1)_{E,\tau} + (U_2,U_1-U_2)_{E,\tau}. \tag{5.21}$$

Lemma 5. *Let assumptions* (2.4), (2.18) *be fulfilled and* η *be given by* (2.31), *where* $d=3$. *Let*

$$\delta = \frac{c_3(0,\mu,\sigma,3)N}{(1-\eta)E^{\sigma/2}} < 1. \tag{5.22}$$

Let $H_\pm(\lambda,p,E)$ *and* $H^0_{E,\tau}(\lambda,p)$ *be defined by* (5.9a), (5.11), $\tau\in]0,1[$. *Then*

$$|H_\pm(\lambda,p,E) - \hat v(p)| \le \frac{\eta N}{(1-\eta)(1+p^2)^{\mu/2}}, \tag{5.23}$$

$$|H_\pm(\lambda,p,E) - H_\pm(\lambda',p,E)| \le \frac{c_6(\alpha,\mu,\sigma,\beta)|\lambda-\lambda'|^\beta N^2}{(1-\eta)^2(1-\delta)(1+p^2)^{\mu/2}}, \tag{5.24}$$

where $\lambda,\lambda'\in\mathcal{T}$, $|\lambda-\lambda'|\le(E-p^2/4)^{-1/2}$, $0<\beta\le\min(\alpha,\sigma,1/2)$, $p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$;

$$H^0_{E,\tau}\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)), \tag{5.25a}$$

$$|||H^0_{E,\tau}|||_{E,\tau,\mu} \le 2^{\mu/2}N + \frac{c_7(\alpha,\mu,\sigma,\beta)N^2}{(1-\eta)^2(1-\delta)E^{\beta/2}},\quad 0<\beta<\min(\alpha,\sigma,1/2). \tag{5.25b}$$

Lemma 5 is proved in Section 11.

Lemma 6. *Let assumptions* (2.4), (2.18) *be fulfilled and* η *be given by* (2.31), *where* $d=3$, $\mu\ge 2$. *Let* $Q_{E,\tau}$ *be defined by* (5.15), $\tau\in]0,1[$. *Then*

$$Q_{E,\tau}\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)), \tag{5.26}$$

$$|||Q_{E,\tau}|||_{E,\tau,\mu_0} \le \frac{3c_5c_4(\mu_0,\tau,E)2^\mu N^2}{(1-\eta)^2(1+2\tau\sqrt{E})^{\mu-\mu_0}},\ 2\le\mu_0\le\mu. \tag{5.27}$$

Lemma 6 follows from (5.15), (4.14), (5.20).

Lemmas 4,5,6 show that, under the assumptions of Proposition 1, the non-linear integral equation (5.10) for unknown $H_{E,\tau}$ can be analyzed for $H^0_{E,\tau}$, $Q_{E,\tau}$, $H_{E,\tau}\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$ using the norm $|||\cdot|||_{E,\tau,\mu_0}$, where $2\le\mu_0\le\mu$.

Consider the equation

$$U = U^0 + M_{E,\tau}(U), \quad E > 0, \quad \tau \in]0,1[, \tag{5.28}$$

for unknown U. Actually, under the assumptions of Proposition 1, we suppose that $U^0 = H^0_{E,\tau} + Q_{E,\tau}$ or consider U^0 as an approximation to $H^0_{E,\tau} + Q_{E,\tau}$.

Lemma 7. *Let $E > 0$, $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $\mu \geq 2$ and $0 < r < (2c_5c_4(\mu,\tau,E))^{-1}$. Let $M_{E,\tau}$ be defined by* (5.12) (*where λ, p are the coordinates of Lemma 1 under assumption* (3.5a)). *Let $U^0 \in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$ and $|||U^0|||_{E,\tau,\mu} \leq r/2$. Then equation* (5.28) *is uniquely solvable for $U \in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, $|||U|||_{E,\tau,\mu} \leq r$, and U can be found by the method of successive approximations, in addition*

$$|||U - M^n_{E,\tau,U^0}(0)|||_{E,\tau,\mu} \leq \frac{(2c_5c_4(\mu,\tau,E)r)^n}{1-2c_5c_4(\mu,\tau,E)r}\frac{r}{2}, \quad n \in \mathbb{N}, \tag{5.29}$$

where M_{E,τ,U^0} denotes the map $U \to U^0 + M_{E,\tau}(U)$.

Lemma 7 is proved in Section 12.

Lemma 8. *Let the assumptions of Lemma 7 be fulfilled. Let also $\tilde U^0 \in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, $|||\tilde U^0|||_{E,\tau,\mu} \leq r/2$, and $\tilde U$ denote the solution of* (5.28) *with U^0 replaced by $\tilde U^0$, where $\tilde U \in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, $|||\tilde U|||_{E,\tau,\mu} \leq r$. Then*

$$|||U - \tilde U|||_{E,\tau,\mu} \leq (1 - 2c_5c_4(\mu,\tau,E)r)^{-1}|||U^0 - \tilde U^0|||_{E,\tau,\mu}. \tag{5.30}$$

Lemma 8 is proved in Section 12.

Proposition 1 and Lemmas 4,5,6,7,8 imply, in particular, the following result.

Proposition 2. *Let assumptions* (2.4), (2.18), (5.22) *be fulfilled, where $d = 3$, $\mu \geq 2$. Let $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $2 \leq \mu_0 \leq \mu$, $0 < \beta < \min(\alpha,\sigma,1/2)$ and*

$$\max(r_1, r_2) < (2c_5c_4(\mu_0,\tau,E))^{-1}, \tag{5.31}$$

where

$$2^{\mu/2}N + \frac{c_7(\alpha,\mu,\sigma,\beta)N^2}{(1-\eta)^2(1-\delta)E^{\beta/2}} + \frac{3c_5c_4(\mu_0,\tau,E)2^\mu N^2}{(1-\eta)^2(1+2\tau\sqrt{E})^{\mu-\mu_0}} = \frac{r_1}{2}, \tag{5.32a}$$

$$\frac{2^{\mu/2}N}{1-\eta} = r_2, \tag{5.32b}$$

and η, δ are given by (2.31) ($d = 3$), (5.22). *Let*

$$\max(r_1, r_2) \leq r < (2c_5c_4(\mu_0,\tau,E))^{-1}.$$

Then

$$|||H_{E,\tau} - \tilde H_{E,\tau}|||_{E,\tau,\mu_0} \leq \frac{3c_5c_4(\mu_0,\tau,E)2^\mu N^2}{(1-2c_5c_4(\mu_0,\tau,E)r)(1-\eta)(1+2\tau\sqrt{E})^{\mu-\mu_0}}, \tag{5.33}$$

where $H_{E,\tau} = H(\lambda, p, E)$ *as a function of* $\lambda \in \mathbb{C}\backslash(\mathcal{T} \cup 0)$ *and* $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$ *(as in Proposition 1) and* $\tilde{H}_{E,\tau}$ *is defined as the solution of*

$$\tilde{H}_{E,\tau} = H^0_{E,\tau} + M_{E,\tau}(\tilde{H}_{E,\tau}), \tag{5.34}$$

where $|||\tilde{H}_{E,\tau}|||_{E,\tau,\mu_0} \le r$.

Using the definitions of η and δ of (2.31), (5.22) we obtain that:

$$\begin{aligned}&\text{if } \ N \le c_8(\alpha,\mu,\mu_0,\sigma,E,\tau), \ \text{ then conditions}\\ &(2.18),(5.22) \ \text{ and } \ (5.31) \ \text{ are fulfilled,}\end{aligned} \tag{5.35}$$

where $0 < \alpha < 1$, $2 \le \mu_0 \le \mu$, $0 < \sigma < 1$, $E \ge 1$, $0 < \tau < 1$, $d = 3$.

Due to (4.12), we have that:

$$c_4(\mu,\tau,E) \le \varepsilon \ \text{ if } 0 < \tau \le \tau(\varepsilon,\mu), \quad E \ge E(\varepsilon,\mu) \tag{5.36}$$

for any arbitrary small $\varepsilon > 0$ and appropriate sufficiently small $\tau(\varepsilon,\mu) \in]0,1[$ and sufficiently great $E(\varepsilon,\mu)$, where $\mu \ge 2$.

Using (5.37) and the definitions of η and δ of (2.31), (5.22) we obtain that:

$$\begin{aligned}&\text{if } \ 0 < \tau \le \tau_1(\alpha,\mu,\mu_0,\sigma,N), \quad E \ge E_1(\alpha,\mu,\mu_0,\sigma,N), \ \text{ then conditions}\\ &(2.18),(5.22) \ \text{ and } \ (5.31) \ \text{ are fulfilled,}\end{aligned} \tag{5.37}$$

where τ_1 and E_1 are appropriate constants such that $\tau_1 \in]0,1[$ is sufficiently small and $E_1 \ge 1$ is sufficiently great, $0 < \alpha < 1$, $2 \le \mu_0 \le \mu$, $0 < \sigma < 1$, $d = 3$.

As a corollary of Propositions 1,2 and property (5.37), we obtain the following result.

Corollary 1. *Let assumptions* (2.4) *be fulfilled, where* $d = 3$, $\mu \ge 2$, *and* $\|\hat{v}\|_{\alpha,\mu} \le N$ *for some* $N > 0$. *Let*

$$0 < \tau \le \tau_1(\alpha,\mu,\mu_0,\sigma,N), \quad E \ge E_1(\alpha,\mu,\mu_0,\sigma,N),$$

where $2 \le \mu_0 \le \mu$, $0 < \sigma < 1$. *Then* $H_\pm(\lambda,p,E)$, $\lambda \in \mathcal{T}$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, *determines* (*via* (5.11), (5.34)) $H(\lambda,p,E)$, $\lambda \in \mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, *up to* $O(E^{-(\mu-\mu_0)/2})$ *in the norm* $|||\cdot|||_{E,\tau,\mu_0}$ *as* $E \to +\infty$ (*due to estimate* (5.33), *where* , *for example*, $r = \max(r_1, r_2)$).

6. Finding $H_\pm$ on $Re\,\Omega_E$ and $H^0_{E,\tau}$ on $(\mathbb{C}\backslash(\mathcal{T}\cup 0)) \times (\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)$ from f on $\mathcal{M}_E$

Under assumptions (2.4), (2.18), due to (2.27), (2.28) we have that

$$\|f\|_{C(\mathcal{M}_E),\mu} \le g_1 N, \tag{6.1}$$

$$\|f\|_{C^\alpha(\mathcal{M}_E),\mu} \le g_2 N, \tag{6.2}$$

$$g_1 = (1-\eta)^{-1}, \quad g_2 = c_2(\mu)(1-\eta)^{-1}, \tag{6.3}$$

where η is given by (2.31).

To find $H_\pm$ on $Re\,\Omega_E$ and $H^0_{E,\tau}$ on $(\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)$ from f on $\mathcal{M}_E$ (for $d=3$) we proceed from (2.20), (5.1) and (5.9a), (5.11). Due to (2.20) we have that

$$h_\gamma(k,\cdot)=f(k,\cdot)+B_\gamma(k)h_\gamma(k,\cdot),\quad \gamma\in\mathbb{S}^{d-1},\quad k\in\mathbb{S}^{d-1}_{\sqrt{E}},\tag{6.4}$$

where the operator $B_\gamma(k)$ is defined by (2.38).

Let us decompose $h_\gamma(k,l)$ as

$$h_\gamma(k,l)=h_\gamma^{(n)}(k,l)+t_\gamma^{(n)}(k,l),\tag{6.5}$$

$$h_\gamma^{(n)}(k,\cdot)=\sum_{j=0}^{n}(B_\gamma(k))^j f(k,\cdot),\tag{6.6}$$

$$t_\gamma^{(n)}(k,\cdot)=(B_\gamma(k))^{n+1}f(k,\cdot)+B_\gamma(k)t_\gamma^{(n)}(k,\cdot),\tag{6.7}$$

where $\gamma\in\mathbb{S}^{d-1}$, $k,l\in\mathbb{S}^{d-1}_{\sqrt{E}}$, $n\in\mathbb{N}\cup 0$.

Further, using (2.8), (5.1), (5.11), (6.5) let us decompose $H_\pm$ and $H^0_{E,\tau}$ as:

$$H_\pm(k,p)=H_\pm^{(n)}(k,p)+T_\pm^{(n)}(k,p),\tag{6.8}$$

$$H_\pm^{(n)}(k,p)=h^{(n)}_{\gamma^\pm(k,p)}(k,k-p),\quad T_\pm^{(n)}(k,p)=t^{(n)}_{\gamma^\pm(k,p)}(k,k-p),\tag{6.9}$$

where $n\in\mathbb{N}\cup 0$, $(k,p)\in Re\,\Omega^1_E$, $|p|\neq 0$, $E>0$;

$$H^0_{E,\tau}(\lambda,p)=H^{0,n}_{E,\tau}(\lambda,p)+T^{0,n}_{E,\tau}(\lambda,p),\tag{6.10}$$

where $n\in\mathbb{N}\cup 0$, $\lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $E>0$, $\tau\in]0,1[$, and $H^{0,n}_{E,\tau}$, $T^{0,n}_{E,\tau}$ are defined by (5.11) with $H_\pm(\zeta,p,E)$ replaced by $H_\pm^{(n)}(k(\zeta,p,E),p)$ and $T_\pm^{(n)}(k(\zeta,p,E),p)$, respectively, where $k(\lambda,p,E)$ is defined in (3.8).

Lemma 9. *Let $d=3$. Let f satisfy* (6.1), (6.2) *and*

$$\begin{aligned}\delta_1&=c_3(0,\mu,\sigma,3)E^{-\sigma/2}g_1N<1,\\ \delta_2&=c_3(\alpha,\mu,\sigma,3)E^{-\sigma/2}g_2N,\end{aligned}\tag{6.11}$$

for some $\alpha\in]0,1[$, $\mu>1$, $\sigma\in]0,\min(1,\mu-1)[$, $E\geq 1$ and some g_1, g_2, $N\in]0,+\infty[$, $g_1<g_2$. Then (6.4), (6.7) *for fixed γ, k and n are uniquely solvable for $h_\gamma(k,\cdot),t_\gamma^{(n)}(k,\cdot)\in C^\alpha(\mathbb{S}^2_{\sqrt{E}})$ (by the method of successive approximations) and the following estimates hold:*

$$|h_\gamma(k,l)|\leq\frac{g_1N}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}},\tag{6.12}$$

$$|h_\gamma(k,l)-h_\gamma(k,l')|\leq\frac{(1+c_2''(\mu)\delta_2)g_2N|l-l'|^\alpha}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}},\tag{6.13}$$

where $\gamma \in \mathbb{S}^2$, $k, l, l' \in \mathbb{S}^2_{\sqrt{E}}$, $|l - l'| \le 1$;

$$|h_\gamma(k,l) - h_{\gamma'}(k',l)| \le \frac{g_2 N |k-k'|^\alpha}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}} + \frac{c_9(\beta,\mu)(g_1N)^2|\gamma-\gamma'|^\beta}{(1-\delta_1)^2(1+|k-l|^2)^{\mu/2}}, \tag{6.14}$$

where $\gamma, \gamma' \in \mathbb{S}^2$, $k, k', l \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma k = \gamma' k' = 0$, $|\gamma - \gamma'| \le 1$, $|k - k'| \le 1$, $0 < \beta < 1/2$;

$$|H_\pm(k(\lambda,p,E),p) - H_\pm(k(\lambda',p,E),p)| \le \frac{c_{10}(\mu)(1+\delta_2)g_2N(E-p^2/4)^{\alpha/2}|\lambda-\lambda'|^\alpha}{(1-\delta_1)(1+p^2)^{\mu/2}} + \frac{c_9(\beta,\mu))(g_1N)^2|\lambda-\lambda'|^\beta}{(1-\delta_1)^2(1+p^2)^{\mu/2}}, \tag{6.15}$$

where $\lambda, \lambda' \in \mathcal{T}$, $|\lambda - \lambda'| \le (E - p^2/4)^{-1/2}$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $k(\lambda, p, E)$ *is defined in* (3.8), $0 < \beta < 1/2$, *and* $H_\pm(k,p)$ *are defined by* (5.1);

$$|H^0_{E,\tau}(\lambda,p)| \le \left(\frac{c_{11} g_1 N (1 + \ln E)}{1-\delta_1} + \frac{c_{12}(\alpha)c_{10}(\mu)(1+\delta_2)g_2N}{1-\delta_1} + \frac{c_{12}(\beta)c_9(\beta,\mu)(g_1N)^2}{(1-\delta_1)^2E^{\beta/2}} \right) \frac{1}{(1+p^2)^{\mu/2}}, \tag{6.16}$$

where $\lambda \in \mathbb{C}\backslash(\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $\tau \in]0,1[$, *and* $H^0_{E,\tau}$ *is defined by* (5.11);

$$|t^{(n)}_\gamma(k,l)| \le \frac{\delta_1^{n+1} g_1 N}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}}, \tag{6.17}$$

$$|t^{(n)}_\gamma(k,l) - t^{(n)}_\gamma(k,l')| \le \frac{c''_2(\mu)\delta_2\delta_1^n g_1 N |l-l'|^\alpha}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}}, \tag{6.18}$$

where $n \in \mathbb{N} \cup 0$, $\gamma \in \mathbb{S}^2$, $k, l, l' \in \mathbb{S}^2_{\sqrt{E}}$, $|l - l'| \le 1$;

$$|t^{(n)}_\gamma(k,l) - t^{(n)}_{\gamma'}(k',l)| \le \frac{\delta_1^{n+1} g_2 N |k-k'|^\alpha}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}} + \frac{c_9(\beta,\mu)\delta_1^n(1+n(1-\delta_1))(g_1N)^2|\gamma-\gamma'|^\beta}{(1-\delta_1)^2(1+|k-l|^2)^{\mu/2}}, \tag{6.19}$$

where $n \in \mathbb{N} \cup 0$, $\gamma, \gamma' \in \mathbb{S}^2$, $k, k', l \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma k = \gamma' k' = 0$, $|\gamma - \gamma'| \le 1$, $|k - k'| \le 1$, $0 < \beta < 1/2$;

$$|T^{(n)}_\pm(k(\lambda,p,E),p) - T^{(n)}_\pm(k(\lambda',p,E),p)| \le \frac{(\delta_1 + c_2(\mu)\delta_2)\delta_1^n g_2 N (E-p^2/4)^{\alpha/2}|\lambda-\lambda'|^\alpha}{(1-\delta_1)(1+p^2)^{\mu/2}} + \frac{c_9(\beta,\mu))\delta_1^n(1+n(1-\delta_1))(g_1N)^2|\lambda-\lambda'|^\beta}{(1-\delta_1)^2(1+p^2)^{\mu/2}}, \tag{6.20}$$

where $n \in \mathbb{N} \cup 0$, $\lambda, \lambda' \in \mathcal{T}$, $|\lambda - \lambda'| \le (E - p^2/4)^{-1/2}$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $k(\lambda, p, E)$ *is defined in* (3.8), $0 < \beta < 1/2$, *and* $T_\pm^{(n)}(k, p)$ *are defined in* (6.9);

$$\begin{aligned}&|T^{0,n}_{E,\tau}(\lambda,p)| \le \Bigg(\frac{c_{11}\delta_1 g_1 N(1+\ln E)}{1-\delta_1} + \\ &\frac{c_{12}(\alpha)(\delta_1 + c_2(\mu)\delta_2)g_2 N}{1-\delta_1} + \frac{c_{12}(\beta)c_9(\beta,\mu)(1+n(1-\delta_1))(g_1N)^2}{(1-\delta_1)^2E^{\beta/2}}\Bigg)\times \\ &\frac{\delta_1^n}{(1+p^2)^{\mu/2}},\end{aligned} \tag{6.21}$$

where $n \in \mathbb{N} \cup 0$, $\lambda \in \mathbb{C} \backslash (\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $\tau \in]0, 1[$, *and* $T^{0,n}_{E,\tau}$ *is defined in* (6.10).

Note that in Lemma 9 we do not suppose that f is the scattering amplitude for some potential. Lemma 9 is proved in Section 11.

Lemma 10. *Let the assumptions of Lemma 9 be fulfilled. Let estimates* (6.1), (6.2) *be also fulfilled for* $\tilde{f}$ *(in place of* f*). Let* $\tilde{h}_\gamma$, $\tilde{H}_\pm$, $\tilde{H}^0_{E,\tau}$ *correspond to* $\tilde{f}$ *as well as* h_γ, $H_\pm$, $H^0_{E,\tau}$ *correspond to* f*. Then:*

$$|h_\gamma(k,l) - \tilde{h}_\gamma(k,l)| \le \frac{\|f-\tilde{f}\|_{C(\mathcal{M}_E),\mu}}{(1-\delta_1)^2(1+|k-l|^2)^{\mu/2}} \quad \gamma \in \mathbb{S}^2, \quad k, l \in \mathbb{S}^2_{\sqrt{E}}; \tag{6.22}$$

$$\begin{aligned}&|(H_\pm - \tilde{H}_\pm)(k(\lambda,p,E),p) - (H_\pm - \tilde{H}_\pm)(k(\lambda',p,E),p)| \le \\ &2\max(1, c_{10}(\mu)(1+\delta_2)(1-\delta_1)g_2N + c_9(\beta,\mu)(g_1N)^2E^{-\beta/2})\times \\ &\frac{E^{\varepsilon\beta/2}|\lambda-\lambda'|^{\varepsilon\beta}(\|f-\tilde{f}\|_{C(\mathcal{M}_E),\mu})^{1-\varepsilon}}{(1-\delta_1)^2(1+p^2)^{\mu/2}},\end{aligned} \tag{6.23}$$

where $\lambda, \lambda' \in \mathcal{T}$, $|\lambda - \lambda'| \le (E - p^2/4)^{-1/2}$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $0 < \beta < min\,(\alpha, 1/2)$, $0 \le \varepsilon \le 1$;

$$\begin{aligned}&|H^0_{E,\tau}(\lambda,p) - \tilde{H}^0_{E,\tau}(\lambda,p)| \le \Bigg(c_{11}(1+\ln E)(\|f-\tilde{f}\|_{C(\mathcal{M}_E),\mu})^{\varepsilon} + \\ &2c_{12}(\varepsilon\beta)\max(1, c_{10}(\mu)(1+\delta_2)(1-\delta_1)g_2N + c_9(\beta,\mu)(g_1N)^2E^{-\beta/2})\Bigg)\times \\ &\frac{(\|f-\tilde{f}\|_{C(\mathcal{M}_E),\mu})^{1-\varepsilon}}{(1-\delta_1)^2(1+p^2)^{\mu/2}},\end{aligned} \tag{6.24}$$

where $\lambda \in \mathbb{C} \backslash (\mathcal{T} \cup 0)$, $p \in \mathcal{B}_{2\sqrt{E}} \backslash \mathcal{L}_\nu$, $0 < \tau < 1$, $0 < \beta < \min(\alpha, 1/2)$, $0 < \varepsilon \le 1$.

Note that in Lemma 10 we do not suppose that f and $\tilde{f}$ are the scattering amplitudes for some potential. Lemma 10 is proved in Section 11.

Let

$$\begin{aligned}&u(s,s_1,s_2) = 1 \quad \text{for} \quad s \in [0, s_1], \\ &u(s,s_1,s_2) = \frac{s_2 - s}{s_2 - s_1} \quad \text{for} \quad s \in [s_1, s_2], \\ &u(s,s_1,s_2) = 0 \quad \text{for} \quad s \in [s_2, +\infty[,\end{aligned} \tag{6.25}$$

where $0 < s_1 < s_2$.

Lemma 11. *Let f satisfy (6.1), (6.2) for some $E > 0$, $\mu > 0$, $\alpha \in]0,1[$ and $g_1, g_2, N \in]0,+\infty[$. Let*

$$\tilde f(k,l) = f(k,l)u(|k-l|, 2\tau_0\sqrt{E}, 2\tau\sqrt{E}), \quad (k,l) \in \mathcal{M}_E, \tag{6.26}$$

where u is defined by (6.25), $0 < \tau_0 < \tau < 1$. Then:

$$\|\tilde f\|_{C(\mathcal{M}_E),\mu} \le g_1 N, \tag{6.27}$$

$$\|\tilde f\|_{C^\alpha(\mathcal{M}_E),\mu} \le \left(g_2 + \frac{g_1}{2(\tau-\tau_0)\sqrt{E}}\right)N, \tag{6.28}$$

$$\|f - \tilde f\|_{C(\mathcal{M}_E),\mu_0} \le \frac{g_1 N}{(1+4\tau_0^2 E)^{(\mu-\mu_0)/2}} \quad \text{for} \quad \mu_0 \in [0,\mu]. \tag{6.29}$$

Lemma 11 is proved in Section 11.

7. Approximate finding $\hat v$ on $\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$ from $\tilde H_{E,\tau}$ on $(\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)$

Consider, first, $H_{E,\tau}$ defined in Proposition 1. Under assumptions (2.4), (2.18), formulas (2.26), (3.8), (3.10), (5.8) imply that

$$\begin{aligned} &H_{E,\tau}(\lambda,p) \to \hat v(p) \quad \text{as} \quad \lambda \to 0,\\ &H_{E,\tau}(\lambda,p) \to \hat v(p) \quad \text{as} \quad \lambda \to \infty, \end{aligned} \tag{7.1}$$

where $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1[$.

Consider now $\tilde H_{E,\tau}$ defined in Proposition 2. Under the assumptions of Proposition 2, the following formulas hold:

$$\tilde H_{E,\tau}(\lambda,p) \to \hat v_+(p,E,\tau) \quad \text{as} \quad \lambda \to 0, \tag{7.2a}$$

$$\tilde H_{E,\tau}(\lambda,p) \to \hat v_-(p,E,\tau) \quad \text{as} \quad \lambda \to \infty, \tag{7.2b}$$

where

$$\begin{aligned} &\hat v_+(p,E,\tau) = \frac{1}{2\pi i}\int_{\mathcal{T}} H_+(\zeta,p,E)\frac{d\zeta}{\zeta} -\\ &\frac{1}{\pi}\iint_{\mathcal{D}_+} (\tilde H_{E,\tau},\tilde H_{E,\tau})_{E,\tau}(\zeta,p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta}, \end{aligned} \tag{7.3a}$$

$$\begin{aligned} &\hat v_-(p,E,\tau) = \frac{1}{2\pi i}\int_{\mathcal{T}} H_-(\zeta,p,E)\frac{d\zeta}{\zeta} +\\ &\frac{1}{\pi}\iint_{\mathcal{D}_-} (\tilde H_{E,\tau},\tilde H_{E,\tau})_{E,\tau}(\zeta,p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta}, \end{aligned} \tag{7.3b}$$

where $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1[$, $H_\pm$ are defined by (5.9a), $(\tilde H_{E,\tau},\tilde H_{E,\tau})_{E,\tau}$ is defined by means of (5.13). Formulas (7.2), (7.3) follow from the definition of $\tilde H_{E,\tau}$ as the solution of (5.28), where $u^0 = H^0_{E,\tau}$, $|||\tilde H_{E,\tau}|||_{E,\tau,\mu} \le r$ (see Proposition 2), formulas (5.11), (5.12) and Lemma 3. Formulas (5.16), (7.1), (7.2) imply that

$$\|\hat v - \hat v_{\pm}(\cdot, E, \tau)\|_{E,\tau,\mu} \le |||H_{E,\tau} - \tilde H_{E,\tau}|||_{E,\tau,\mu}, \tag{7.4}$$

where

$$\|w\|_{E,\tau,\mu} = \sup_{p\in\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu} (1+|p|)^\mu |w(p)|, \tag{7.5}$$

$E \ge 1$, $\tau \in]0,1[$, $\mu < 0$, w is a test function. Under the assumptions of Proposition 2 (or under the assumptions of Corollary 1), formulas (5.33), (7.4) imply that $\hat v$ on $\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu$ can be approximately determined from $\tilde H_{E,\tau}$ on $(\mathbb{C}\setminus(\mathcal{T}\cup 0))\times\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu$ as $\hat v_{\pm}(\cdot, E,\tau)$ of (7.2), (7.3) and

$$\|\hat v - \hat v_{\pm}(\cdot, E, \tau)\|_{E,\tau,\mu_0} = O(E^{-(\mu-\mu_0)/2}) \ \text{ as } \ E \to +\infty. \tag{7.6}$$

8. Approximate finding $\hat v$ on $\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu$ from f on $\mathcal{M}_E$ for $d = 3$

In this section we summarize our method (developed in Sections 3,4,5,6,7) for approximate finding $\hat v$ on $\mathcal{B}_{2\tau\sqrt{E}}$ from f on $\mathcal{M}_E$.

Consider $\tau_1(\alpha,\mu,\mu_0,\sigma,N)$ and $E_1(\alpha,\mu,\mu_0,\sigma,N)$ of (5.37), where $0<\alpha<1$, $2\le\mu_0\le\mu$, $0<\sigma<1$, $N>0$. Let g_1, g_2 be some fixed numbers such that

$$\begin{aligned} &g_1 \ge \left(1 - E^{-\sigma/2}c_1(\alpha,\mu,\sigma,3)N\right)^{-1},\\ &g_2 \ge c_2(\mu)(1 - E^{-\sigma/2}c_1(\alpha,\mu,\sigma,3)N)^{-1}, \quad g_1 < g_2,\\ &\text{for} \ \ E \ge E_1(\alpha,\mu,\mu_0,\sigma,N). \end{aligned} \tag{8.1}$$

Consider $E_2(\mu,\sigma,N,g_1) \ge 1$ such that

$$\delta_1 < 1 \ \text{ for } \ E \ge E_2(\mu,\sigma,N,g_1), \tag{8.2}$$

where δ_1 is defined in (6.11), $2\le\mu$, $0<\sigma<1$, $N>0$.

Theorem 1. *Let $\hat v$ satisfy* (2.4), *where $d=3$, $\mu\ge 2$ and $\|\hat v\|_{\alpha,\mu}\le N$ for some $N>0$. Let*

$$0<\tau<\tau_1(\alpha,\mu,\mu_0,\sigma,N), \quad E \ge \max\left(E_1(\alpha,\mu,\mu_0,\sigma,N),\ E_2(\mu,\sigma,N,g_1)\right), \tag{8.3}$$

where $2\le\mu_0\le\mu$, $0<\sigma<1$ and g_1 satisfies (8.1). *Let f be the scattering amplitude for equation* (1.1) *and $\hat v_\pm(p,E,\tau)$ be determined from f on $\mathcal{M}_E$ via the following (stable) reconstruction procedure:*

$$\begin{aligned} &f \ \text{ on } \ \mathcal{M}_E \overset{(2.20)}{\to} h_\gamma(k,l),\ (k,l)\in\mathcal{M}_E,\ \gamma\in\mathbb{S}^2,\ \gamma k = 0\\ &\overset{(2.8),(5.1),(5.9a)}{\to} H_\pm(\lambda,p,E),\ \lambda\in\mathcal{T},\ p\in\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu,\\ &\overset{(5.11),(5.25b)}{\to} H^0_{E,\tau}(\lambda,p),\ \lambda\in\mathbb{C}\setminus(\mathcal{T}\cup 0),\ p\in\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu,\\ &\overset{(5.34)}{\to} \tilde H_{E,\tau}(\lambda,p),\ \lambda\in\mathbb{C}\setminus(\mathcal{T}\cup 0),\ p\in\mathcal{B}_{2\tau\sqrt{E}}\setminus\mathcal{L}_\nu, \overset{(7.2),(7.3)}{\to} \hat v_\pm(p,E,\tau),\ p\in\mathcal{B}_{2\tau\sqrt{E}}, \end{aligned} \tag{8.4}$$

where integral equations (2.20), (5.34) *are solved by the method of successive approximations. Then*

$$\|\hat{v} - \hat{v}_\pm(\cdot, E, \tau)\|_{E,\tau,\mu_0} \le \frac{3c_5 c_4(\mu_0, \tau, E) 2^\mu N^2}{(1 - 2c_5 c_4(\mu_0, \tau, E) r)(1-\eta)(1 + 2\tau\sqrt{E})^{\mu-\mu_0}}, \tag{8.5}$$

where $\|\cdot\|_{E,\tau,\mu_0}$ *is defined by* (7.5), $\eta = E^{-\sigma/2} c_1(\alpha, \mu, \sigma, 3)N$, $r = \max(r_1, r_2)$, *where* r_1, r_2 *are defined by* (5.32), *and thus*

$$\|\hat{v} - \hat{v}_\pm(\cdot, E, \tau)\|_{E,\tau,\mu_0} = O(E^{-(\mu-\mu_0)/2}) \quad as \quad E \to +\infty \tag{8.6}$$

at fixed $\alpha, \mu, \mu_0, \tau, N$.

Theorem 1 follows from estimates (6.1)-(6.3) and Lemma 9 of Section 6, Lemmas 5,7,8, Proposition 2 and property (5.37) of Section 5, formulas (7.2)-(7.4) of Section 7 and the definition of $E_2(\alpha, \mu, \sigma, N, g_2)$.

Note that for finding $H^0_{E,\tau}$ from $H_\pm$ in (8.4) we mention also estimate (5.25b), in addition to formula (5.11). This means that: (1) we determine $H^0_{E,\tau}$ from $H_\pm$ by (5.11); (2) we redefine $H^0_{E,\tau}$ by the formulas:

$$\begin{aligned} &H^0_{E,\tau}(\lambda, p) \ \to \ H^0_{E,\tau}(\lambda, p) \ \text{ if} \\ &|H^0_{E,\tau}(\lambda, p)| \le \left(2^{\mu/2} N + \frac{c_7(\alpha, \mu, \sigma, \beta) N^2}{(1-\eta)^2 (1-\delta) E^{\beta/2}}\right) \frac{1}{(1+|p|)^\mu}, \end{aligned} \tag{8.7a}$$

$$\begin{aligned} &H^0_{E,\tau}(\lambda, p) \to \left(2^{\mu/2} N + \frac{c_7(\alpha, \mu, \sigma, \beta) N^2}{(1-\eta)^2 (1-\delta) E^{\beta/2}}\right) \frac{1}{(1+|p|)^\mu} \frac{H^0_{E,\tau}(\lambda, p)}{|H^0_{E,\tau}(\lambda, p)|} \ \text{ if} \\ &|H^0_{E,\tau}(\lambda, p)| > \left(2^{\mu/2} N + \frac{c_7(\alpha, \mu, \sigma, \beta) N^2}{(1-\eta)^2 (1-\delta) E^{\beta/2}}\right) \frac{1}{(1+|p|)^\mu}, \end{aligned} \tag{8.7b}$$

where η is defined by (2.31), $d = 3$, δ is defined by (5.22), $0 < \beta < \min(\alpha, \sigma, 1/2)$. If, under the assumptions of Theorem 1, the scattering amplitude f is given with no errors and all calculations based on (2.20), (2.8), (5.1), (5.9a), (5.11) of (8.4) are fulfilled with no errors, then estimate (5.25b) is fulfilled automatically. However, if some of these errors are present, then estimate (5.25b) taken into account according to (8.7) permits to improve the stability of the reconstruction procedure (8.4).

Theorem 2. *Let* g_1, g_2 satisfy (8.1). *Let* $\hat{v}$, N, τ, E, α, μ, μ_0, σ *satisfy the assumptions of Theorem 1. Let* f *be the scattering amplitude for equation* (1.1) *and, as in Theorem 1,* $\hat{v}_\pm(\cdot, E, \tau)$ *be determined from* f *on* $\mathcal{M}_E$ *via* (8.4). *Let* $\tilde{f}$ *be an approximation to* f, *where estimates* (6.1), (6.2) *are valid also for* $\tilde{f}$ *in place of* f. *Let* $\tilde{\hat{v}}_\pm(\cdot, E, \tau)$ *be determined from* $\tilde{f}$ *on* $\mathcal{M}_E$ *via* (8.4) (*including* (8.7)) *with* $\tilde{f}$, $\tilde{h}_\gamma$, $\tilde{H}_\pm$, $\tilde{H}^0_{E,\tau}$, $\tilde{\tilde{H}}_{E,\tau}$, $\tilde{\hat{v}}(\cdot, E, \tau)$ *in place of* f, h_γ, $H_\pm$, $H^0_{E,\tau}$, $\tilde{H}_{E,\tau}$, $\hat{v}(\cdot, E, \tau)$, *respectively. Then*

$$\begin{aligned} &\|\hat{v}_\pm(\cdot, E, \tau) - \tilde{\hat{v}}_\pm(\cdot, E, \tau)\|_{E,\tau,\mu_0} \le \big(c_{11}(1 + \ln E)(\|f - \tilde{f}\|_{C(\mathcal{M}_E),\mu})^\varepsilon + \\ &2c_{12}(\varepsilon\beta) \max(1, c_{10}(\mu)(1+\delta_2)(1-\delta_1) g_2 N + c_9(\beta, \mu)(g_1 N)^2 E^{-\beta/2})\big) \times \\ &\frac{(\|f - \tilde{f}\|_{C(\mathcal{M}_E),\mu})^{1-\varepsilon}}{(1-\delta_1)^2 (1 - 2c_5 c_4(\mu_0, \tau, E) r)}, \quad r = \max(r_1, r_2), \end{aligned} \tag{8.8}$$

where $0 < \varepsilon \le 1$, $0 < \beta < \min(\alpha, 1/2)$, δ_1, δ_2 *are defined in* (6.11), r_1, r_2 *are defined by* (5.32).

Theorem 2 follows from Lemma 10 of Section 6, Lemma 8 of Section 5 and the following formula

$$\|\hat v_\pm(\cdot, E,\tau) - \tilde{\hat v}_\pm(\cdot, E,\tau)\|_{E,\tau,\mu_0} \le |||\tilde H_{E,\tau} - \tilde{\tilde H}_{E,\tau}|||_{E,\tau,\mu_0}. \tag{8.9}$$

Consider $\mathcal{M}_{E,\tau}$ defined by (1.14). Consider the function $\tilde f$ defined by (6.25) on $\mathcal{M}_E$, where f is the scattering amplitude for equation (1.1). Note that $\tilde f \equiv 0$ on $\mathcal{M}_E \backslash \mathcal{M}_{E,\tau}$ and $\tilde f$ is completely determined from f on $\mathcal{M}_{E,\tau}$. As a corollary of Theorems 1 and 2 of this section, estimates (6.1)-(6.3) and Lemma 11 of Section 6 we obtain the following results.

Corollary 2. *Let* $\hat v$, N, τ, E, α, μ, μ_0, σ *satisfy the assumptions of Theorem 1. Let* f *be the scattering amplitude for equation* (1.1) *and* $\tilde f$ *on* $\mathcal{M}_E$ *be defined in terms of* f *on* $\mathcal{M}_{E,\tau}$ *by* (6.25). *Let, further,* $\tilde{\hat v}(\cdot, E,\tau)$ *be reconstructed from* $\tilde f$ *on* $\mathcal{M}_E$ *via* (8.4) (*including* (8.7)) *as in Theorem 2. Then*

$$\|\hat v - \tilde{\hat v}(\cdot, E,\tau)\|_{E,\tau,\mu_0} = O\left(\frac{1}{E^{(1-\varepsilon)(\mu-\mu_0)/2}}\right) \ \text{as} \ \ E \to +\infty \tag{8.10}$$

for fixed $\varepsilon \in]0,1[$ *and* α, μ, μ_0, τ, τ_0, N.

Let us remind that if v satisfies (1.2), then $\hat v$ satisfies (2.3). Using this remark and Theorem 1 we obtain the following result.

Corollary 3. *Let* v satisfy (1.2), *where* $d = 3$, $n > 3$. *Then:*

(1) $\hat v$ *satisfies the assumptions of Theorem 1 for* $\mu = n$ *and some* $\alpha \in]0,1[$ *and* $N > 0$;

(2) *the scattering amplitude* f *on* $\mathcal{M}_E$ *determines* v *on* $\mathbb{R}^3$ *up to* $O(E^{-(n-3-\varepsilon)/2})$ *in the uniform norm as* $E \to +\infty$ *for any fixed arbitrary small* $\varepsilon > 0$ *via* (8.4) (*where* τ *and* E *satisfy* (8.3) *with* α, μ *and* N *of the item* (1), $\mu_0 = 3 + \varepsilon$ *and some* $\sigma \in]0,1[$) *and the formula*

$$v(x) = v^\pm_{appr}(x, E,\tau) + v^\pm_{err}(x, E,\tau), \tag{8.11}$$

where

$$\begin{aligned} v^\pm_{appr}(x, E,\tau) &= \int\limits_{\mathcal{B}_{2\tau\sqrt{E}}} e^{-ipx}\hat v_\pm(p, E,\tau)dp, \\ v^\pm_{err}(x, E,\tau) &= \int\limits_{\mathcal{B}_{2\tau\sqrt{E}}} e^{-ipx}(\hat v(p) - \hat v_\pm(p, E,\tau))dp + \int\limits_{\mathbb{R}^3\backslash\mathcal{B}_{2\tau\sqrt{E}}} \hat v(p)dp. \end{aligned} \tag{8.12}$$

9. Proof of Lemma 2

First of all, for $d = 3$ we write equation (2.22) as

$$\frac{\partial}{\partial \bar k_j}H(k,p) = -\frac{\pi}{2}\int\limits_{\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}} \xi_j H(k,-\xi)H(k+\xi,p+\xi)\frac{ds}{|Im\, k|\,|Re\, k|} \tag{9.1}$$

for $j=1,2,3$, $k\in\mathbb{C}^3\backslash\mathbb{R}^3$, $k^2=E$, $p\in\mathbb{R}^3$, where ds is arc-length measure on the circle $\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}$.

Taking into account (3.10), (3.8) we have that

$$\begin{gathered}\frac{\partial}{\partial\bar\lambda}H(k(\lambda,p,E),p)=\sum_{j=1}^{3}\frac{\partial\bar k_j}{\partial\bar\lambda}\frac{\partial H(k,p)}{\partial\bar k_j}=\\ \sum_{j=1}^{3}\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\theta_j+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\omega_j\right)\frac{\partial H(k,p)}{\partial\bar k_j},\quad \lambda\in\mathbb{C}\backslash 0,\ p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu,\end{gathered}\tag{9.2}$$

where $k=k(\lambda,p,E)$, $\kappa_1=\kappa_1(\lambda,p,E)$, $\kappa_2=\kappa_2(\lambda,p,E)$, $\theta=\theta(p)$, $\omega=\omega(p)$, see (3.8), (3.4).

Formulas (9.1), (9.2) imply that

$$\begin{gathered}\frac{\partial}{\partial\bar\lambda}H(k(\lambda,p,E),p)=-\frac{\pi}{2}\int\limits_{\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}}\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\theta\xi+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\omega\xi\right)\times\\ H(k,-\xi)H(k+\xi,p+\xi)\frac{ds}{|Im\,k|\,|Re\,k|},\ \lambda\in\mathbb{C}\backslash 0,\ p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu,\end{gathered}\tag{9.3}$$

where $k=k(\lambda,p,E)$, $\kappa_1=\kappa_1(\lambda,p,E)$, $\kappa_2=\kappa_2(\lambda,p,E)$, $\theta=\theta(p)$, $\omega=\omega(p)$, $\tau\in]0,1]$.

Let the circle $\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}$ (where $k=k(\lambda,p,E)$) be parametrized by $\varphi\in]-\pi,\pi[$ according to (4.2). In the parametrization (4.2) the following formula holds:

$$ds=|Re\,k|d\varphi,\tag{9.4}$$

where ds is arc-length measure on $\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}$.

Further,

$$\begin{gathered}k^\perp\overset{(4.3),(3.8)}{=}|Im\,k|^{-1}(Im\,\kappa_1\,\theta+Im\,\kappa_2\,\omega)\times(Re\,\kappa_1\,\theta+Re\,\kappa_2\,\omega+p/2)=\\ |Im\,k|^{-1}(Im\,\kappa_1\,\theta\times Re\,\kappa_1\,\theta+Im\,\kappa_1\,\theta\times Re\,\kappa_2\,\omega+Im\,\kappa_1\,\theta\times p/2+\\ Im\,\kappa_2\,\omega\times Re\,\kappa_1\,\theta+Im\,\kappa_2\,\omega\times Re\,\kappa_2\,\omega+Im\,\kappa_2\,\omega\times p/2).\end{gathered}\tag{9.5}$$

Formula (9.5) and the formulas

$$\theta\times\omega\overset{(3.4),(3.5a)}{=}\hat p,\quad \theta\times\hat p\overset{(3.4),(3.5a)}{=}-\omega,\quad \omega\times\hat p\overset{(3.4),(3.5a)}{=}\theta,\quad \theta\times\theta=0,\quad \omega\times\omega=0,\tag{9.6}$$

where $\hat p=p/|p|$, imply that

$$k^\perp=|Im\,k|^{-1}(Im\,\kappa_1\,Re\,\kappa_2-Im\,\kappa_2\,Re\,\kappa_1)\hat p+Im\,\kappa_2\,\frac{|p|}{2}\theta-Im\,\kappa_1\,\frac{|p|}{2}\omega).\tag{9.7}$$

Formulas (3.4), (3.8), (4.2), (9.7) imply that

$$\begin{gathered}\theta\xi=Re\,\kappa_1(\cos\varphi-1)+(2|Im\,k|)^{-1}Im\,\kappa_2|p|\sin\varphi,\\ \omega\xi=Re\,\kappa_2(\cos\varphi-1)-(2|Im\,k|)^{-1}Im\,\kappa_1|p|\sin\varphi.\end{gathered}\tag{9.8}$$

Due to (9.8) we have that

$$\begin{aligned}&\frac{\partial\bar\kappa_1}{\partial\bar\lambda}\theta\xi+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}\omega\xi=\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}Re\,\kappa_1+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}Re\,\kappa_2\right)(\cos\varphi-1)+\\&\frac{|p|}{2|Im\,k|}\left(\frac{\partial\bar\kappa_1}{\partial\bar\lambda}Im\,\kappa_2-\frac{\partial\bar\kappa_2}{\partial\bar\lambda}Im\,\kappa_1\right)\sin\varphi.\end{aligned}\tag{9.9}$$

The definition of κ_1, κ_2 (see (3.8)) implies that

$$\frac{\partial\bar\kappa_1}{\partial\bar\lambda}=\big(1-\frac{1}{\bar\lambda^2}\big)\frac{(E-p^2/4)^{1/2}}{2},\quad\frac{\partial\bar\kappa_2}{\partial\bar\lambda}=\big(1+\frac{1}{\bar\lambda^2}\big)\frac{i(E-p^2/4)^{1/2}}{2},\tag{9.10}$$

$$\begin{aligned}&Re\,\kappa_1=\big(\lambda+\bar\lambda+\frac{1}{\lambda}+\frac{1}{\bar\lambda}\big)\frac{(E-p^2/4)^{1/2}}{4},\quad Im\,\kappa_1=\big(\lambda-\bar\lambda+\frac{1}{\lambda}-\frac{1}{\bar\lambda}\big)\frac{(E-p^2/4)^{1/2}}{4i},\\&Re\,\kappa_2=\big(\frac{1}{\lambda}-\frac{1}{\bar\lambda}-\lambda+\bar\lambda\big)\frac{i(E-p^2/4)^{1/2}}{4},\quad Im\,\kappa_2=\big(\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\lambda-\bar\lambda\big)\frac{(E-p^2/4)^{1/2}}{4},\end{aligned}\tag{9.11}$$

where $\lambda\in\mathbb{C}\backslash 0$, $p\in\mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$. Due to (9.10), (9.11) we have that

$$\begin{aligned}&\frac{\partial\bar\kappa_1}{\partial\bar\lambda}Re\,\kappa_1+\frac{\partial\bar\kappa_2}{\partial\bar\lambda}Re\,\kappa_2=\\&\frac{E-p^2/4}{8}\Big(\big(1-\frac{1}{\bar\lambda^2}\big)\big(\lambda+\bar\lambda+\frac{1}{\lambda}+\frac{1}{\bar\lambda}\big)-\big(1+\frac{1}{\bar\lambda^2}\big)\big(\frac{1}{\lambda}-\frac{1}{\bar\lambda}-\lambda+\bar\lambda\big)\Big)=\\&\frac{E-p^2/4}{8}\big(\lambda+\bar\lambda+\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\frac{\lambda}{\bar\lambda^2}-\frac{1}{\bar\lambda}-\frac{1}{\bar\lambda^2\lambda}-\frac{1}{\bar\lambda^3}-\frac{1}{\lambda}+\frac{1}{\bar\lambda}+\\&\lambda-\bar\lambda-\frac{1}{\bar\lambda^2\lambda}+\frac{1}{\bar\lambda^3}+\frac{\lambda}{\bar\lambda^2}-\frac{1}{\bar\lambda}\big)=\\&\frac{E-p^2/4}{4}\big(\lambda-\frac{1}{\bar\lambda^2\lambda}\big)=\frac{E-p^2/4}{4}\frac{|\lambda|^4-1}{|\lambda|^2\bar\lambda},\end{aligned}\tag{9.12}$$

$$\begin{aligned}&\frac{\partial\bar\kappa_1}{\partial\bar\lambda}Im\,\kappa_2-\frac{\partial\bar\kappa_2}{\partial\bar\lambda}Im\,\kappa_1=\\&\frac{E-p^2/4}{8}\Big(\big(1-\frac{1}{\bar\lambda^2}\big)\big(\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\lambda-\bar\lambda\big)-\big(1+\frac{1}{\bar\lambda^2}\big)\big(\lambda-\bar\lambda+\frac{1}{\lambda}-\frac{1}{\bar\lambda}\big)\Big)=\\&\frac{E-p^2/4}{8}\big(\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\lambda-\bar\lambda-\frac{1}{\bar\lambda^2\lambda}-\frac{1}{\bar\lambda^3}+\frac{\lambda}{\bar\lambda^2}+\frac{1}{\bar\lambda}-\\&\lambda+\bar\lambda-\frac{1}{\lambda}+\frac{1}{\bar\lambda}-\frac{\lambda}{\bar\lambda^2}+\frac{1}{\bar\lambda}-\frac{1}{\bar\lambda^2\lambda}+\frac{1}{\bar\lambda^3}\big)=\\&\frac{E-p^2/4}{4}\big(-\lambda+\frac{2}{\bar\lambda}-\frac{1}{\bar\lambda^2\lambda}\big)=-\frac{E-p^2/4}{4}\frac{(|\lambda|^2-1)^2}{|\lambda|^2\bar\lambda}.\end{aligned}\tag{9.13}$$

The $\bar\partial$- equation (4.1) follows from (9.3), (9.4), (9.9), (9.12), (9.13), (3.9).
Lemma 2 is proved.

10. Proof of Lemma 3

Let us show, first, that

$$\{U_1, U_2\}(\cdot,\cdot,E) \in L^\infty_{local}((\mathbb{C}\backslash(\mathcal{T}\cup 0)) \times (\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)),\ \tau \in]0,1[. \tag{10.1}$$

Property (10.1) follows from formula (4.6), the properties

$$U_1(k, -\xi(k,\varphi)) \in L^\infty((\Sigma_E\backslash Re\,\Sigma_E)\times[0,2\pi]) \ \text{ (as a function of } \ k,\varphi), \tag{10.2}$$

$$\begin{aligned}&U_2(k+\xi(k,\varphi), p+\xi(k,\varphi)) \in L^\infty((\Omega_E\backslash Re\,\Omega_E)\times[0,2\pi])\\ &\text{(as a function of } \ k,p,\varphi),\end{aligned} \tag{10.3}$$

where

$$\Sigma_E = \{k\in\mathbb{C}^3:\ k^2=E\},\quad Re\,\Sigma_E = \{k\in\mathbb{R}^3:\ k^2=E\}, \tag{10.4}$$

$$\xi(k,\varphi) = Re\,k(\cos\varphi - 1) + k^\perp\sin\varphi,\quad k^\perp = \frac{Im\,k\times Re\,k}{|Im\,k|}, \tag{10.5}$$

and Lemma 1. In turn, (10.2) follows from $U_1\in L^\infty(\Omega_E\backslash Re\,\Omega_E)$, definition (2.11) ($d=3$) and the fact that $p=-\xi(k,\varphi)$, $\varphi\in[0,2\pi]$, is a parametrization of the set $\{p\in\mathbb{R}^3:\ p^2 = 2kp\}$, $k\in\Sigma_E\backslash Re\,\Sigma_E$. To prove (10.3) consider

$$\begin{aligned}&\Omega'_E = \{k\in\mathbb{C}^3,\ l\in\mathbb{C}^3:\ k^2=l^2=E,\ Im\,k = Im\,l\},\\ &Re\,\Omega'_E = \{k\in\mathbb{R}^3,\ l\in\mathbb{C}^3:\ k^2=l^2=E\}.\end{aligned} \tag{10.6}$$

Note that

$$\begin{aligned}&\Omega'_E\backslash Re\,\Omega'_E \approx \Omega_E\backslash Re\,\Omega_E,\\ &(k,l)\in\Omega'_E\backslash Re\,\Omega'_E \Rightarrow (k,k-l)\in\Omega_E\backslash Re\,\Omega_E,\\ &(k,p)\in\Omega_E\backslash Re\,\Omega_E \Rightarrow (k,k-p)\in\Omega'_E\backslash Re\,\Omega'_E.\end{aligned} \tag{10.7}$$

Consider

$$u_2(k,l) = U_2(k,k-l),\quad (k,l)\in\Omega'_E\backslash Re\,\Omega'_E. \tag{10.8}$$

The property $U_2\in L^\infty(\Omega_E\backslash Re\,\Omega_E)$ is equivalent to the property $u_2\in L^\infty(\Omega'_E\backslash Re\,\Omega'_E)$. Property (10.3) is equivalent to the property

$$u_2(k+\xi(k,\varphi), l) \in L^\infty((\Omega'_E\backslash Re\,\Omega'_E)\times[0,2\pi]) \ \text{ (as a function of } \ k,l,\varphi). \tag{10.9}$$

Property (10.9) follows from the property

$$\begin{aligned}&u_2(\zeta(l,\psi,\varphi)+iIm\,l, l)\in L^\infty((\Sigma_E\backslash Re\,\Sigma_E)\times[0,2\pi]\times[0,2\pi]),\\ &\text{(as a function of } \ l,\psi,\varphi),\end{aligned} \tag{10.10}$$

where

$$\zeta(l,\psi,\varphi) = Re\,l\,\cos(\varphi-\psi) + l^\perp\sin(\varphi-\psi),\quad l^\perp = \frac{Im\,l\times Re\,l}{|Im\,l|}. \tag{10.11}$$

Note that $k = \zeta(l,\psi,\varphi)$, $\varphi \in [0,2\pi]$, at fixed $\psi \in [0,2\pi]$ is a parametrization of the set $S_l = \{k \in \mathbb{C}^3 :\ k^2 = l^2,\ Im\,k = Im\,l\}$, $l \in \Sigma_E \backslash Re\,\Sigma_E$. In turn, (10.10) follows from $u_2 \in L^\infty(\Omega'_E \backslash Re\,\Omega'_E)$, definition (10.6) and the aforementioned fact concerning the parametrization of S_l. Thus, properties (10.10), (10.9), (10.3) are proved. This completes the proof of (10.1).

Let us prove now (4.11).

We have that

$$\{U_1,U_2\}(\lambda,p,E) = \{U_1,U_2\}_1(\lambda,p,E) + \{U_1,U_2\}_2(\lambda,p,E), \tag{10.12}$$

where

$$\{U_1,U_2\}_1(\lambda,p,E) = -\frac{\pi}{4}(E-p^2/4)^{1/2}\frac{sgn\,(|\lambda|^2-1)(|\lambda|^2+1)}{\bar\lambda|\lambda|}\{U_1,U_2\}_3(\lambda,p,E), \tag{10.13a}$$

$$\begin{gathered}\{U_1,U_2\}_3(\lambda,p,E) = \int_{-\pi}^{\pi}(\cos\varphi - 1)\times\\ U_1(k(\lambda,p,E),-\xi(\lambda,p,E,\varphi))U_2(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))d\varphi,\end{gathered} \tag{10.13b}$$

$$\{U_1,U_2\}_2(\lambda,p,E) = \frac{\pi}{4}\frac{|p|}{\bar\lambda}\{U_1,U_2\}_4(\lambda,p,E), \tag{10.14a}$$

$$\begin{gathered}\{U_1,U_2\}_4(\lambda,p,E) = \int_{-\pi}^{\pi}\sin\varphi\times\\ U_1(k(\lambda,p,E),-\xi(\lambda,p,E,\varphi))U_2(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))d\varphi,\end{gathered} \tag{10.14b}$$

$\lambda \in \mathbb{C}\backslash(\mathcal{T}\cup 0)$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$, $\tau \in]0,1[$.

Formulas (4.2), (4.3) imply that

$$|\xi|^2 = |Re\,k|^2((\cos\varphi-1)^2 + (\sin\varphi)^2) = 4|Re\,k|^2(\sin(\varphi/2))^2, \tag{10.15}$$

where $\xi = \xi(\lambda,p,E,\varphi)$, $k = k(\lambda,p,E)$.

The relation $p^2 = 2k(\lambda,p,E)p$, $p \in \mathbb{R}^3$, implies that

$$p = -Re\,k(\lambda,p,E)(\cos\psi - 1) - k^\perp(\lambda,p,E)\sin\psi \tag{10.16}$$

for some $\psi = \psi(\lambda,p,E) \in [-\pi,\pi]$, where $k^\perp(\lambda,p,E)$ is defined by (4.3). Formulas (4.2), (4.3), (10.16) imply that

$$\begin{gathered}|p+\xi|^2 = |Re\,k|^2((\cos\varphi-\cos\psi)^2 + (\sin\varphi-\sin\psi)^2) = 4|Re\,k|^2\big(\sin\frac{\varphi-\psi}{2}\big)^2,\\ |p|^2 = 4|Re\,k|^2\big(\sin\frac{\psi}{2}\big)^2,\end{gathered} \tag{10.17}$$

where $\xi = \xi(\lambda,p,E,\varphi)$, $k = k(\lambda,p,E)$, $\psi = \psi(\lambda,p,E)$.

Using the assumptions of Lemma 3 and formulas (10.13b), (10.14b), (10.15), (10.17) we obtain that

$$|\{U_1, U_2\}_3(\lambda, p, E)| \le \int\limits_{-\pi}^{\pi} \frac{(1-\cos\varphi)|||U_1|||_{E,\mu}|||U_2|||_{E,\mu} d\varphi}{(1+2r|\sin(\varphi/2)|)^{\mu}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\mu}}, \tag{10.18}$$

$$|\{U_1, U_2\}_4(\lambda, p, E)| \le \int\limits_{-\pi}^{\pi} \frac{|\sin\varphi|\,|||U_1|||_{E,\mu}|||U_2|||_{E,\mu} d\varphi}{(1+2r|\sin(\varphi/2)|)^{\mu}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\mu}}, \tag{10.19}$$

where $r = Re\, k(\lambda, p, E)$, $\psi = \psi(\lambda, p, E)$.

Consider

$$A(r,\psi,\alpha,\beta) = \int\limits_{-\pi}^{\pi} \frac{(1-\cos\varphi) d\varphi}{(1+2r|\sin(\varphi/2)|)^{\alpha}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\beta}}, \tag{10.20a}$$

$$B(r,\psi,\alpha,\beta) = \int\limits_{-\pi}^{\pi} \frac{|\sin\varphi|\, d\varphi}{(1+2r|\sin(\varphi/2)|)^{\alpha}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\beta}}, \tag{10.20b}$$

where $r \ge 0$, $\psi \in [-\pi, \pi]$, $\alpha \ge 2$, $\beta \ge 2$. Due to (10.18)-(10.20) we have that

$$|\{U_1, U_2\}_3(\lambda, p, E)| \le |||U_1|||_{E,\mu}|||U_2|||_{E,\mu} A(r,\psi,\mu,\mu), \tag{10.21a}$$
$$|\{U_1, U_2\}_4(\lambda, p, E)| \le |||U_1|||_{E,\mu}|||U_2|||_{E,\mu} B(r,\psi,\mu,\mu), \tag{10.21b}$$

where $r = Re\, k(\lambda, p, E)$, $\psi = \psi(\lambda, p, E)$.

Lemma 12. *Let $r \ge 0$, $\psi \in [-\pi, \pi]$, $\rho = 2r|\sin(\psi/2)|$, $\alpha \ge 2$, $\beta \ge 2$. Then*

$$A(r,\psi,\alpha,\beta) \le \sum_{j=1}^{4} A_j(r,\psi,\alpha,\beta), \tag{10.22}$$

$$A_1(r,\psi,\alpha,\beta) \le \min\left(\frac{\rho^3}{6r^3}, \frac{\rho}{r^3}\right)\frac{1}{(1+\rho/2)^{\beta}}, \tag{10.23}$$

$$A_2(r,\psi,\alpha,\beta) \le \frac{\rho^3}{r^3}\,\frac{1}{(1+\rho/2)^{\alpha+1}}, \tag{10.24}$$

$$A_3(r,\psi,\alpha,\beta) \le \frac{4\rho^3}{r^3}\,\frac{1}{(1+\rho)^{\alpha}(1+\rho/2)}, \tag{10.25}$$

$$A_4(r,\psi,\alpha,\beta) \le \left(\frac{3}{1+r^2} + \frac{2\pi}{(1+\sqrt{2}r)^{\alpha}}\right)\frac{1}{(1+\rho/2)^{\beta}}, \tag{10.26}$$

$$B(r,\psi,\alpha,\beta) \le \sum_{j=1}^{4} B_j(r,\psi,\alpha,\beta), \tag{10.27}$$

$$B_1(r,\psi,\alpha,\beta) \le \min\left(\frac{\rho^2}{2r^2}, \frac{\sqrt{2}\rho}{r^2}\right)\frac{1}{(1+\rho/2)^{\beta}}, \tag{10.28}$$

$$B_2(r,\psi,\alpha,\beta) \le \frac{2\rho^2}{r^2}\,\frac{1}{(1+\rho/2)^{\alpha+1}}, \tag{10.29}$$

$$B_3(r,\psi,\alpha,\beta) \le \frac{4\rho^2}{r^2}\,\frac{1}{(1+\rho)^{\alpha}(1+\rho/2)}, \tag{10.30}$$

$$B_4(r,\psi,\alpha,\beta) \le \left(\frac{5}{1+r} + \frac{3}{(1+\sqrt{2}r)^{\alpha}}\right)\frac{1}{(1+\rho/2)^{\beta}}. \tag{10.31}$$

Proof of Lemma 12. Note that

$$A(r,\psi,\alpha,\beta) = A(r,-\psi,\alpha,\beta), \quad B(r,\psi,\alpha,\beta) = B(r,-\psi,\alpha,\beta). \tag{10.32}$$

Therefore, it is sufficient to prove Lemma 12 for $\psi \in [0,\pi]$. Let

$$\begin{aligned}
W_1(r,\psi,\alpha,\beta,\varphi) &= \frac{1-\cos\varphi}{(1+2r|\sin(\varphi/2)|)^{\alpha}(1+2r|\sin\left(\frac{\varphi-\psi}{2}\right)|)^{\beta}},\\
W_2(r,\psi,\alpha,\beta,\varphi) &= \frac{\sin\varphi}{(1+2r|\sin(\varphi/2)|)^{\alpha}(1+2r|\sin\left(\frac{\varphi-\psi}{2}\right)|)^{\beta}}.
\end{aligned} \tag{10.33}$$

For $\psi \in [0,\pi]$ we have that:

$$A(r,\psi,\alpha,\beta) \le 2\int\limits_0^{\pi} W_1(r,\psi,\alpha,\beta,\varphi)d\varphi = \sum_{j=1}^{4} A_j(r,\psi,\alpha,\beta), \tag{10.34}$$

where

$$A_1(r,\psi,\alpha,\beta) = 2\int\limits_0^{\psi/2} W_1(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.35}$$

$$A_2(r,\psi,\alpha,\beta) = 2\int\limits_{\psi/2}^{\psi} W_1(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.36}$$

$$A_3(r,\psi,\alpha,\beta) = 2\int\limits_{\psi}^{\min(3\psi/2,\pi)} W_1(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.37}$$

$$A_4(r,\psi,\alpha,\beta) = 2\int\limits_{\min(3\psi/2,\pi)}^{\pi} W_1(r,\psi,\alpha,\beta,\varphi)d\varphi; \tag{10.38}$$

$$B(r,\psi,\alpha,\beta) \le 2\int\limits_0^{\pi} W_2(r,\psi,\alpha,\beta,\varphi)d\varphi = \sum_{j=1}^{4} B_j(r,\psi,\alpha,\beta), \tag{10.39}$$

where

$$B_1(r,\psi,\alpha,\beta) = 2\int\limits_0^{\psi/2} W_2(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.40}$$

$$B_2(r,\psi,\alpha,\beta) = 2\int\limits_{\psi/2}^{\psi} W_2(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.41}$$

$$B_3(r,\psi,\alpha,\beta) = 2\int\limits_{\psi}^{\min(3\psi/2,\pi)} W_2(r,\psi,\alpha,\beta,\varphi)d\varphi, \tag{10.42}$$

$$B_4(r,\psi,\alpha,\beta) = 2\int\limits_{\min(3\psi/2,\pi)}^{\pi} W_2(r,\psi,\alpha,\beta,\varphi)d\varphi. \tag{10.43}$$

To prove Lemma 12 it remains to prove estimates (10.23)-(10.26), (10.28)-(10.31) for $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ defined by (10.35)-(10.38), (10.40)-(10.43) for $\psi \in [0,\pi]$. Note that in these proofs given below we largely use the following formulas

$$\begin{aligned} &\rho = 2r\sin(\psi/2) = 4r\sin(\psi/4)\,\cos(\psi/4),\\ &\rho/2 \le 2r\sin(\psi/4), \quad \sin(\psi/4) \le \frac{\rho}{2\sqrt{2}r}, \end{aligned} \tag{10.44}$$

where $\psi \in [0,\pi]$.

Proof of estimate (10.23) *for* A_1 *of* (10.35). We have that

$$\begin{aligned} &A_1 = \int\limits_0^{\psi/2} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}(1+2r|\sin\left(\frac{\varphi-\psi}{2}\right)|)^{\beta}} \le\\ &\frac{1}{(1+2r\sin(\psi/4))^{\beta}}\int\limits_0^{\psi/2}\frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}}, \end{aligned} \tag{10.45}$$

$$\frac{1}{(1+2r\sin(\psi/4))^{\beta}} \overset{(10.44)}{\le} \frac{1}{(1+\rho/2)^{\beta}}, \tag{10.46}$$

$$\begin{aligned} &\int\limits_0^{\psi/2}\frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \le \int\limits_0^{\psi/2} 4(\sin(\varphi/2))^2 d\varphi \le 8\sqrt{2}\int\limits_0^{\psi/2}(\sin(\varphi/2))^2 d\sin(\varphi/2) \le\\ &\frac{8\sqrt{2}}{3}(\sin(\psi/4))^3 \overset{(10.44)}{\le} \frac{8\sqrt{2}}{3}\left(\frac{\rho\sqrt{2}}{4r}\right)^3 = \frac{1}{6}\left(\frac{\rho}{r}\right)^3, \end{aligned} \tag{10.47a}$$

$$\int\limits_0^{\psi/2} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^\alpha} \le \int\limits_0^{\psi/2} \frac{4d\varphi}{4r^2(1+2r\sin(\varphi/2))^{\alpha-2}} \le$$
$$\frac{\psi}{2}\frac{1}{r^2} \le \frac{\pi}{2}\sin(\psi/2)\frac{1}{r^2} = \frac{\pi}{4}\frac{\rho}{r^3}, \qquad (10.47b)$$

where $\psi \in [0,\pi]$, $\alpha \ge 2$, $\beta \ge 2$. Estimate (10.23) for A_1 of (10.35) follows from (10.45)-(10.47).

Proof of estimates (10.24) *for* A_2 *of* (10.36). We have that

$$A_2 = \int\limits_{\psi/2}^{\psi} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^\alpha(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^\beta} \le$$
$$\frac{4(\sin(\psi/2))^2}{(1+2r\sin(\psi/4))^\alpha}\int\limits_0^{\psi/2} \frac{d\varphi}{(1+2r\sin(\varphi/2))^\beta}, \qquad (10.48)$$

$$\frac{4(\sin(\psi/2))^2}{(1+2r\sin(\psi/4))^\alpha} \overset{(10.44)}{\le} \frac{(\rho/r)^2}{(1+\rho/2)^\alpha}, \qquad (10.49)$$

$$\int\limits_0^{\psi/2} \frac{d\varphi}{(1+2r\sin(\varphi/2))^\beta} = \int\limits_0^{\psi/2} \frac{(2/\cos(\varphi/2))d\sin(\varphi/2)}{(1+2r\sin(\varphi/2))^\beta} \le$$
$$2\sqrt{2}\int\limits_0^{\sin(\psi/4)} \frac{dt}{(1+2rt)^\beta} = \frac{\sqrt{2}}{r(\beta-1)}\left(1-\frac{1}{(1+2r\sin(\psi/4))^{\beta-1}}\right) \le \qquad (10.50)$$
$$\frac{\sqrt{2}2r\sin(\psi/4)}{r(1+2r\sin(\psi/4))} \overset{(10.44)}{\le} \frac{\rho}{r(1+\rho/2)},$$

where $\psi \in [0,\pi]$, $\alpha \ge 2$, $\beta \ge 2$. Estimate (10.24) for A_2 of (10.36) follows from (10.48)-(10.50).

Proof of estimate (10.25) *for* A_3 *of* (10.37). We have that

$$
\begin{aligned}
A_3 = &\int\limits_{\psi}^{\min(\pi,3\psi/2)} \frac{16\big(\sin(\varphi/4)\cos(\varphi/4)\big)^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}\big(1+2r\sin(\frac{\varphi-\psi}{2})\big)^{\beta}} \le \\
&\frac{16(\sin(3\psi/8)^2}{(1+2r\sin(\psi/2))^{\alpha}} \int\limits_{\psi}^{\min(\pi,3\psi/2)} \frac{d\varphi}{\big(1+2r\sin(\frac{\varphi-\psi}{2})\big)^{\beta}} \le \\
&\frac{16(\sin(\psi/2)^2}{(1+2r\sin(\psi/2))^{\alpha}} \int\limits_{0}^{\psi/2} \frac{d\varphi}{(1+2r\sin(\varphi/2))^{\beta}} \le \\
&\overset{(10.44),(10.50)}{\le} \frac{4\rho^3}{r^3(1+\rho)^{\alpha}(1+\rho/2)},
\end{aligned}
\tag{10.51}
$$

where $\psi \in [0,\pi]$, $\alpha \ge 2$, $\beta \ge 2$. Estimate (10.25) for A_3 of (10.37) is proved.

Proof of estimate (10.26) *for* A_4 *of* (10.38). We have that

$$
\begin{aligned}
A_4 = &\int\limits_{\min(\pi,3\psi/2)}^{\pi} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}\big(1+2r\sin\big(\frac{\varphi-\psi}{2}\big)\big)^{\beta}} \le \\
&\frac{1}{(1+2r\sin(\psi/4))^{\beta}} \int\limits_{0}^{\pi} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \le \\
&\frac{1}{(1+\rho/2)^{\beta}} \Big(\int\limits_{0}^{\pi/2} + \int\limits_{\pi/2}^{\pi} \Big) \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}},
\end{aligned}
\tag{10.52}
$$

$$
\int\limits_{0}^{\pi/2} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \overset{(10.47a)}{\le} \frac{8\sqrt{2}}{3}(\sin(\pi/4))^3 = \frac{4}{3}, \tag{10.53a}
$$

$$
\int\limits_{0}^{\pi/2} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \overset{(10.47b)}{\le} \frac{\pi}{2r^2}, \tag{10.53b}
$$

$$
\int\limits_{\pi/2}^{\pi} \frac{4(\sin(\varphi/2))^2 d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \le \frac{2\pi}{(1+2r\sin(\pi/4))^{\alpha}}, \tag{10.54}
$$

where $\psi \in [0,\pi]$, $\alpha \ge 2$, $\beta \ge 2$. Formulas (10.52)-(10.54) imply that

$$
A_4 \le \frac{4}{3}\min\big(1, \frac{5}{4r^2}\big) + \frac{2\pi}{(1+\sqrt{2}r)^{\alpha}}. \tag{10.55}
$$

Estimate (10.26) for A_4 of (10.38) follows from (10.55) and the inequality

$$\frac{1+c}{1+s} \geq \min\big(1, \frac{c}{s}\big),\ \ c \geq 0,\ \ s \geq 0. \tag{10.56}$$

Proof of estimate (10.28) *for* B_1 *of* (10.40). We have that

$$\begin{gathered}
B_1 = \int\limits_0^{\psi/2} \frac{2\sin(\varphi)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\beta}} \leq \\
\frac{1}{(1+2r\sin(\psi/4))^{\beta}} \int\limits_0^{\psi/2} \frac{2\sin(\varphi)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \leq \\
\frac{1}{(1+\rho/2)^{\beta}} \int\limits_0^{\psi/2} \frac{4\sin(\varphi/2)\cos(\varphi/2)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} = \frac{1}{(1+\rho/2)^{\beta}} \int\limits_0^{\sin(\psi/4)} \frac{8tdt}{(1+2rt)^{\alpha}},
\end{gathered} \tag{10.57}$$

$$\int\limits_0^{\sin(\psi/4)} \frac{8tdt}{(1+2rt)^{\alpha}} \leq \int\limits_0^{\sin(\psi/4)} 8tdt = 4(\sin(\psi/4))^2 \overset{(10.44)}{\leq} \frac{\rho^2}{2r^2}, \tag{10.58a}$$

$$\int\limits_0^{\sin(\psi/4)} \frac{8tdt}{(1+2rt)^{\alpha}} \leq \frac{4\sin(\psi/4)}{r} \overset{(10.44)}{\leq} \frac{\sqrt{2}\rho}{r^2}, \tag{10.58b}$$

where $\psi \in [0, 2\pi]$, $\alpha \geq 2$, $\beta \geq 2$. Estimate (10.28) for B_1 of (10.40) follows from (10.59), (10.60).

Proof of estimate (10.29) *for* B_2 *of* (10.41). We have that

$$\begin{gathered}
B_2 = \int\limits_{\psi/2}^{\psi} \frac{4\sin(\varphi/2)\ \cos(\varphi/2)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}(1+2r|\sin\big(\frac{\varphi-\psi}{2}\big)|)^{\beta}} \leq \\
\frac{4\sin(\psi/2)}{(1+2r\sin(\psi/4))^{\alpha}} \int\limits_0^{\psi/2} \frac{d\varphi}{(1+2r\sin(\varphi/2))^{\beta}} \\
\overset{(10.44),(10.50)}{\leq} \frac{2\rho^2}{r^2(1+\rho/2)^{\alpha+1}},
\end{gathered} \tag{10.59}$$

where $\psi \in [0, \pi]$, $\alpha \geq 2$, $\beta \geq 2$. Estimate (10.29) for B_2 of (10.41) is proved.

Proof of estimate (10.30) *for* B_3 *of* (10.42). We have that

$$
\begin{aligned}
&B_3 = \int\limits_{\psi}^{\min(\pi,3\psi/2)} \frac{8\sin(\varphi/4)\,\cos(\varphi/4)\,\cos(\varphi/2)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}(1+2r\sin\big(\frac{\varphi-\psi}{2}\big))^{\beta}} \le \\
&\frac{8\sin(3\psi/8)}{(1+2r\sin(\psi/2))^{\alpha}} \int\limits_{\psi}^{\min(\pi,3\psi/2)} \frac{d\varphi}{(1+2r\sin\big(\frac{\varphi-\psi}{2}\big))^{\beta}} \le \\
&\frac{8\sin(\psi/2)}{(1+2r\sin(\psi/2))^{\alpha}} \int\limits_{0}^{\psi/2} \frac{d\varphi}{(1+2r\sin(\varphi/2))^{\beta}} \\
&\overset{(10.44),(10.50)}{\le} \frac{4\rho^2}{r^2(1+\rho)^{\alpha}(1+\rho/2)},
\end{aligned} \tag{10.60}
$$

where $\psi\in[0,\pi]$, $\alpha\ge 2$, $\beta\ge 2$. Estimate (10.30) for B_3 of (10.42) is proved.

Proof of estimate (10.31) *for* B_4 *of* (10.43). We have that

$$
\begin{aligned}
&B_4 = \int\limits_{\min(\pi,3\psi/2)}^{\pi} \frac{2\sin(\varphi)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}(1+2r\sin\big(\frac{\varphi-\psi}{2}\big))^{\beta}} \le \\
&\frac{1}{(1+2r\sin(\psi/4))^{\beta}} \int\limits_{0}^{\pi} \frac{4\sin(\varphi/2)\cos(\varphi/2)d\varphi}{(1+2r\sin(\varphi/2))^{\alpha}} \le \\
&\frac{1}{(1+\rho/2)^{\beta}} \Bigg(\int\limits_{0}^{\pi/2} + \int\limits_{\pi/2}^{\pi}\Bigg) \frac{8\sin(\varphi/2)d\sin(\varphi/2)}{(1+2r\sin(\varphi/2))^{\alpha}} = \\
&\frac{1}{(1+\rho/2)^{\beta}} \Bigg(\int\limits_{0}^{\sin(\pi/4)} + \int\limits_{\sin(\pi/4)}^{\sin(\pi/2)}\Bigg) \frac{8tdt}{(1+2rt)^{\alpha}},
\end{aligned} \tag{10.61}
$$

$$
\int\limits_{0}^{\sin(\pi/4)} \frac{8tdt}{(1+2rt)^{\alpha}} \overset{(10.58)}{\le} 2\min\big(1,\frac{\sqrt{2}}{r}\big) \overset{(10.56)}{\le} \frac{2(1+\sqrt{2})}{1+r}, \tag{10.62}
$$

$$
\int\limits_{\sin(\pi/4)}^{\sin(\pi/2)} \frac{8tdt}{(1+2rt)^{\alpha}} \le \frac{4(2-\sqrt{2})}{(1+\sqrt{2}r)^{\alpha}}, \tag{10.63}
$$

where $\psi\in[0,\pi]$, $\alpha\ge 2$, $\beta\ge 2$. Estimate (10.31) for B_4 of (10.43) follows from (10.61)-(10.63).

Lemma 12 is proved.

Lemma 13. *Let*

$$r = r(\lambda,\rho,E) = ((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)^{1/2}/2, \qquad (10.64)$$
$$|\sin(\psi/2)| = \rho/(2r),$$

where $\lambda\in\mathbb{C}\backslash 0$, $\rho\in[0,2\tau\sqrt{E}]$, $E>0$, $\tau\in]0,1[$, $\psi\in[-\pi,\pi]$. *Let*

$$z=(1-\tau^2)/(4\tau^2). \qquad (10.65)$$

Then:

$$\frac{(E-\rho^2/4)^{1/2}(|\lambda|^2+1)}{|\lambda|^2}A_1 \le \frac{4|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho/2)^{\beta}}, \qquad (10.66)$$

$$\frac{(E-\rho^2/4)^{1/2}(|\lambda|^2+1)}{|\lambda|^2}A_2 \le \frac{16|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho/2)^{\alpha}}, \qquad (10.67)$$

$$\frac{(E-\rho^2/4)^{1/2}(|\lambda|^2+1)}{|\lambda|^2}A_3 \le \frac{64|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho)^{\alpha}}, \qquad (10.68)$$

$$\frac{(E-\rho^2/4)^{1/2}(|\lambda|^2+1)}{|\lambda|^2}A_4 \le \frac{4\sqrt{2}(3+\pi)}{(|\lambda|^2+1)\sqrt{E}\min(1,2\sqrt{z})(1+\rho/2)^{\beta}}, \qquad (10.69)$$

$$\frac{\rho B_1}{|\lambda|} \le \frac{4\sqrt{2}|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho/2)^{\beta}}, \qquad (10.70)$$

$$\frac{\rho B_2}{|\lambda|} \le \frac{16|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho/2)^{\alpha}}, \qquad (10.71)$$

$$\frac{\rho B_3}{|\lambda|} \le \frac{32|\lambda|}{(|\lambda|^2+1)^2 z(1+\rho)^{\alpha}}, \qquad (10.72)$$

$$\frac{\rho B_4}{|\lambda|} \le \frac{15}{(|\lambda|^2+1)\sqrt{z}(1+\rho/2)^{\beta}}, \qquad (10.73)$$

where $A_j=A_j(r,|\psi|,\alpha,\beta)$, $B_j=B_j(r,|\psi|,\alpha,\beta)$ *are the same that in Lemma 12,* $j=1,2,3,4$, $\alpha\ge 2$, $\beta\ge 2$.

Proof of Lemma 13. The property $\rho\in[0,2\tau\sqrt{E}]$ and formula (10.65) imply that

$$E-\rho^2/4 \ge z\rho^2. \qquad (10.74)$$

Further, proceeding from (10.23)-(10.26), (10.28)-(10.31), (10.64) and (10.74) we prove separately each of estimates (10.67)-(10.74).

Proof of (10.66). Note that

$$\frac{(E-\rho^2/4)^{1/2}\rho^3}{6r^3} \overset{(10.64)}{=} \frac{8(E-\rho^2/4)^{1/2}\rho^3}{6((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)^{3/2}} \le$$
$$\frac{4\rho^3}{3(|\lambda|+|\lambda|^{-1})((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)} \overset{(10.74)}{\le} \frac{4\rho}{3z(|\lambda|+|\lambda|^{-1})^3}, \qquad (10.75)$$

$$\begin{aligned}&\frac{(E-\rho^2/4)^{1/2}\rho}{r^3}\overset{(10.64)}{=}\frac{8(E-\rho^2/4)^{1/2}\rho}{((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)^{3/2}}\le\\&\frac{8\rho}{(|\lambda|+|\lambda|^{-1})((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)}\overset{(10.74)}{\le}\frac{8}{\rho z(|\lambda|+|\lambda|^{-1})^3},\end{aligned}\tag{10.76}$$

$$\begin{aligned}&(E-\rho^2/4)^{1/2}\min\Big(\frac{\rho^3}{6r^3},\frac{\rho}{r^3}\Big)\overset{(10.75)-(10.76)}{\le}\\&\frac{8\min(\rho/6,1/\rho)}{z(|\lambda|+|\lambda|^{-1})^3}\le\frac{8}{\sqrt{6}(|\lambda|+|\lambda|^{-1})^3}.\end{aligned}\tag{10.77}$$

Estimate (10.66) follows from (10.23), (10.77).

Proof of (10.67). Estimate (10.67) follows from (10.24) and (10.75).

Proof of (10.68). Estimate (10.68) follows from (10.25) and (10.75).

Proof of (10.69). Note that

$$\begin{aligned}&(E-\rho^2/4)^{1/2}\Big(\frac{3}{1+r^2}+\frac{2\pi}{(1+\sqrt{2}r)^\alpha}\Big)\le\frac{(E-\rho^2/4)^{1/2}(3+\pi)}{r^2}\overset{(10.64)}{\le}\\&\frac{4(3+\pi)}{(|\lambda|+|\lambda|^{-1})((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)^{1/2}}\le\frac{4(3+\pi)}{(E-\rho^2/4)^{1/2}(|\lambda|+|\lambda|^{-1})^2}.\end{aligned}\tag{10.78}$$

In addition,

$$\begin{aligned}&\rho^2\le 2E\Rightarrow E-\rho^2/4\ge E/2;\\&\rho^2\ge 2E,\ E-\rho^2/2\ge z\rho^2\ \ \text{(see }\ (10.74))\Rightarrow E-\rho^2/2\ge 2zE.\end{aligned}\tag{10.79}$$

Thus,

$$E-\rho^2/2\ge\min(E/2,2zE)\ \ \text{for}\ \ \rho\in[0,2\tau\sqrt{E}].\tag{10.80}$$

Estimate (10.69) follows from (10.26), (10.78), (10.80).

Proof of (10.70). Note that

$$\frac{\rho^3}{2r^2}\overset{(10.64)}{=}\frac{2\rho^3}{(E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2}\overset{(10.74)}{\le}\frac{2\rho}{z(|\lambda|+|\lambda|^{-1})^2},\tag{10.81}$$

$$\frac{\sqrt{2}\rho^2}{r^2}\overset{(10.64)}{=}\frac{4\sqrt{2}\rho^2}{(E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2}\overset{(10.74)}{\le}\frac{4\sqrt{2}}{z(|\lambda|+|\lambda|^{-1})^2}.\tag{10.82}$$

Estimate (10.70) follows from (10.28), (10.81), (10.82).

Proof of (10.71). Estimate (10.71) follows from (10.29), (10.82).

Proof of (10.72). Estimate (10.72) follows from (10.30), (10.82).

Proof of (10.73). Note that

$$\rho\left(\frac{3}{1+r}+\frac{3}{(1+\sqrt{2}r)^{\alpha}}\right)\le\frac{(5+3/\sqrt{2})\rho}{r}\overset{(10.64)}{=}\\ \frac{(10+3\sqrt{2})\rho}{((E-\rho^2/4)(|\lambda|+|\lambda|^{-1})^2+\rho^2)^{1/2}}\le\frac{15}{\sqrt{z}(|\lambda|+|\lambda|^{-1})}. \tag{10.83}$$

Estimate (10.73) follows from (10.31), (10.83).

Lemma 13 is proved.

Estimate (4.11) follows from (10.12)-(10.14), (10.21), (10.22) and Lemma 13.

Property (4.10) follows from (10.1), (4.11).

Lemma 3 is proved.

11. Proofs of Lemmas 5, 9, 10, 11

Proof of Lemma 5. Estimate (5.23) follows from (2.8), (2.29a), (5.1), (5.9a). To prove estimate (5.24) we proceed from estimates (2.27), (2.29) and equation (2.20). Due to (2.8), (2.29), we have, in particular, that

$$|h_\gamma(k,l)|\le\frac{N}{(1-\eta)(1+|k-l|^2)^{\mu/2}}, \tag{11.1a}$$

$$|h_\gamma(k,l)-h_\gamma(k,l')-\hat v(k-l)+\hat v(k-l')|\le\frac{c_2''(\mu)\eta N|l-l'|^{\alpha}}{(1-\eta)(1+|k-l|^2)^{\mu/2}}, \tag{11.1b}$$

where $k,l,l'\in\mathbb{S}^2_{\sqrt{E}}$, $\gamma\in\mathbb{S}^2$, $|l-l'|\le 1$ (and η is given by (2.31)).

Consider formulas (6.5)-(6.7) for $n=0$. Note that

$$h^{(0)}_\gamma(k,l)=f(k,l),\quad h_\gamma(k,l)=f(k,l)+t^{(0)}_\gamma(k,l), \tag{11.2}$$

where $\gamma\in\mathbb{S}^2$, $k,l\in\mathbb{S}^2_{\sqrt{E}}$. The following estimate holds:

$$|t^{(0)}_\gamma(k,l)-t^{(0)}_{\gamma'}(k',l)|\le\frac{N^2(c_2(\mu)c_3(0,\mu,\sigma,3)E^{-\sigma/2}|k-k'|^{\alpha}+c_{13}(\beta,\mu)|\gamma-\gamma'|^{\beta})}{(1-\eta)^2(1-\delta)(1+|k-l|^2)^{\mu/2}}, \tag{11.3}$$

where $k,k',l\in\mathbb{S}^2_{\sqrt{E}}$, $\gamma,\gamma'\in\mathbb{S}^2$, $\gamma k=\gamma'k'=0$, $|\gamma-\gamma'|\le 1$, $|k-k'|\le 1$, $0<\beta\le 1/2$ (and η, δ are given by (2.31), (5.22)).

Proof of (11.3). Using (6.7) for $n=0$ we obtain that

$$t^{(0)}_\gamma(k,\cdot)-t^{(0)}_{\gamma'}(k',\cdot)=B_\gamma(k)f(k,\cdot)-B_{\gamma'}(k')f(k',\cdot)+B_\gamma(k)t^{(0)}_\gamma(k,\cdot)-\\ B_{\gamma'}(k')t^{(0)}_{\gamma'}(k',\cdot), \tag{11.4}$$

$$t^{(0)}_\gamma(k,\cdot)-t^{(0)}_{\gamma'}(k',\cdot)=B_\gamma(k)(f(k,\cdot)-f(k',\cdot))+(B_\gamma(k)-B_{\gamma'}(k'))\times\\ (f(k',\cdot)+t^{(0)}_{\gamma'}(k',\cdot))+B_\gamma(k)(t^{(0)}_\gamma(k,\cdot)-t^{(0)}_{\gamma'}(k',\cdot)), \tag{11.5}$$

where $k, k' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$. We consider (11.5) as a linear integral equation for $t^{(0)}_{\gamma}(k,\cdot) - t^{(0)}_{\gamma'}(k',\cdot)$. In addition, the following estimates hold:

$$|B_\gamma(k)(f(k,\cdot) - f(k',\cdot))(l)| \le \frac{\delta c_2(\mu) N^2 |k-k'|^\alpha}{(1-\eta)^2(1+|k-l|^2)^{\mu/2}}, \tag{11.6}$$

$$|(B_\gamma(k) - B_{\gamma'}(k'))(f(k',\cdot) + t^{(0)}_{\gamma'}(k',\cdot))(l)| \le \frac{c_2'(\mu) c_{13}(\beta,\mu) N^2 |\gamma-\gamma'|^\beta}{(1-\eta)^2(1+|k-l|^2)^{\mu/2}}, \tag{11.7}$$

where $k, k', l \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $\gamma k = \gamma' k' = 0$, $|\gamma - \gamma'| \le 1$, $|k-k'| \le 1$, $0 < \beta < 1/2$, $E \ge 1$. Estimate (11.6) follows from (2.39) (for $\beta = 0$), (2.27a), (2.28a), (5.22). Estimate (11.7) follows from (11.1a), (11.2) and the following lemma:

Lemma 14. *Let $B_\gamma(k)$ and Λ_k be defined by* (2.38), (2.40). *Then:*

$$\begin{aligned}&\|\Lambda_k^\mu (B_\gamma(k) - B_{\gamma'}(k'))\Lambda_k^{-\mu} u\|_{C(\mathbb{S}^2_{\sqrt{E}})} \le c_{13}(\beta,\mu) E^{\beta-1/2} \|f\|_{C(\mathcal{M}_E),\mu} \times \\ &\|u\|_{C(\mathbb{S}^2_{\sqrt{E}})} |\gamma-\gamma'|^\beta,\end{aligned} \tag{11.8}$$

where $k, k' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $\gamma k = \gamma' k' = 0$, $|\gamma - \gamma'| \le 1$, $\mu \ge 2$, $0 < \beta \le 1/2$, $E \ge 1$.

Proof of Lemma 14. From (2.38) it follows that

$$(B_\gamma(k) - B_{\gamma'}(k'))U(l) = \frac{\pi i}{\sqrt{E}} \int\limits_{\mathbb{S}^2_{\sqrt{E}}} U(m)(\chi(m\gamma) - \chi(m\gamma')) f(m,l)\, dm, \tag{11.9}$$

$$\begin{aligned}&|(B_\gamma(k) - B_{\gamma'}(k'))\Lambda_k^{-\mu} u(l)| \le \frac{\pi}{\sqrt{E}} \|f\|_{C(\mathcal{M}_E),\mu} \|u\|_{C(\mathbb{S}^2_{\sqrt{E}})} \times \\ &\int\limits_{\mathbb{S}^2_{\sqrt{E}}} \frac{|\chi(m\gamma) - \chi(m\gamma')|\, dm}{(1+|k-m|^2)^{\mu/2}(1+|m-l|^2)^{\mu/2}},\end{aligned} \tag{11.10}$$

where $k, k' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $\gamma k = \gamma' k' = 0$. Consider

$$\begin{aligned}&\mathcal{D}_{k,l} = \{m \in \mathbb{S}^2_{\sqrt{E}} : \ |m-k| < |m-l|\}, \\ &\mathcal{D}_{l,k} = \{m \in \mathbb{S}^2_{\sqrt{E}} : \ |m-k| > |m-l|\}, \quad k, l \in \mathbb{S}^2_{\sqrt{E}}.\end{aligned} \tag{11.11}$$

Note that

$$|m-l| > |k-l|/2 \ \text{ for } \ m \in \mathcal{D}_{k,l}, \quad |m-k| > |k-l|/2 \ \text{ for } \ m \in \mathcal{D}_{l,k}. \tag{11.12}$$

We have that

$$
\begin{gathered}
\frac{1}{\sqrt{E}}\int\limits_{\mathbb{S}^2_{\sqrt{E}}}\frac{|\chi(m\gamma)-\chi(m\gamma')|dm}{(1+|k-m|^2)^{\mu/2}(1+|m-l|^2)^{\mu/2}}\overset{(11.12)}{\le}\frac{1}{\sqrt{E}(1+|k-l|^2/4)^{\mu/2}}\times\\
\Bigg(\int\limits_{\mathcal{D}_{k,l}}\frac{|\chi(m\gamma)-\chi(m\gamma')|dm}{(1+|k-m|^2)^{\mu/2}}+\int\limits_{\mathcal{D}_{l,k}}\frac{|\chi(m\gamma)-\chi(m\gamma')|dm}{(1+|m-l|^2)^{\mu/2}}\Bigg)\le\\
\frac{1}{\sqrt{E}(1+|k-l|^2/4)^{\mu/2}}\Bigg(\int\limits_{\mathbb{S}^2_{\sqrt{E}}}\frac{|\chi(m\gamma)-\chi(m\gamma')|dm}{(1+|k-m|^2)^{\mu/2}}+\int\limits_{\mathbb{S}^2_{\sqrt{E}}}\frac{|\chi(m\gamma)-\chi(m\gamma')|dm}{(1+|m-l|^2)^{\mu/2}}\Bigg)\le\\
\frac{\sqrt{E}}{(1+|k-l|^2/4)^{\mu/2}}\Bigg(\int\limits_{\mathbb{S}^2_{\sqrt{E}}}\frac{|\chi(n\gamma)-\chi(n\gamma')|dn}{(1+E|n-\hat{k}|^2)^{\mu/2}}+\int\limits_{\mathbb{S}^2_{\sqrt{E}}}\frac{|\chi(n\gamma)-\chi(n\gamma')|dn}{(1+E|n-\hat{l}|^2)^{\mu/2}}\Bigg),
\end{gathered}
\tag{11.13}
$$

where $k,l\in\mathbb{S}^2_{\sqrt{E}}$, $\hat{k}=k/\sqrt{E}$, $\hat{l}=l/\sqrt{E}$. Further, by the Hölder inequality we have that

$$
\begin{gathered}
\int\limits_{\mathbb{S}^2}\frac{|\chi(n\gamma)-\chi(n\gamma')|dn}{(1+E|n-\omega|^2)^{\mu/2}}\le\Bigg(\int\limits_{\mathbb{S}^2}|\chi(n\gamma)-\chi(n\gamma')|^p dn\Bigg)^{1/p}\times\\
\Bigg(\int\limits_{\mathbb{S}^2}\frac{dn}{(1+E|n-\omega|^2)^{p\mu/(2(p-1))}}\Bigg)^{(p-1)/p},\quad \gamma,\gamma',\omega\in\mathbb{S}^2,\ p\in]1,+\infty[.
\end{gathered}
\tag{11.14}
$$

In addition,

$$
\int\limits_{\mathbb{S}^2}\frac{dn}{(1+E|n-\omega|^2)^{p/(p-1)}}\le\frac{c_{14}(p)}{E},\quad p\in]1,+\infty[,
\tag{11.15}
$$

$$
\begin{gathered}
\int\limits_{\mathbb{S}^2}|\chi(n\gamma)-\chi(n\gamma')|^p dn=\int\limits_{\mathbb{S}^2}|\chi(n\gamma)-\chi(n\gamma')|dn=\\
2\int\limits_0^{\pi}\sin(\psi)\int\limits_0^{\varphi(\gamma,\gamma')}d\varphi d\psi=4\varphi(\gamma,\gamma')\le\frac{4\pi}{3}|\gamma-\gamma'|,
\end{gathered}
\tag{11.16}
$$

where $|\gamma-\gamma'|\le 1$ and $\varphi(\gamma,\gamma')$ is (absolute value of) the angle between γ and γ'. Note that in (11.16) we have used the following: (1) the property that $\chi(n\gamma)-\chi(n\gamma')$ takes values in the set $\{-1,0,1\}$; (2) Euclidean basis e_1, e_2, e_3, where $e_1=\gamma$, $e_3=\frac{\gamma\times\gamma'}{|\gamma\times\gamma'|}$, $e_2=-e_1\times e_3$, and the standard spherical coordinates related with this basis; (3) the formulas $|\gamma-\gamma'|/2=\sin(\varphi(\gamma,\gamma')/2)$, $1/2=\sin(\pi/6)$, $(\pi/3)\sin(\varphi/2)\ge\varphi/2$ for $\varphi\in[0,\pi/3]$.

Estimate (11.8) follows from (11.10), (11.13)-(11.16), where $p=1/\beta$. Lemma 14 is proved. Thus, estimate (11.7) is also proved.

Estimate (11.3) follows from (11.5)-(11.7) and (2.39) (for $\beta=0$), (5.22). This completes the proof of (11.3).

Using (11.2) we obtain that

$$
\begin{aligned}
&h_\gamma(k,l) - h_{\gamma'}(k',l') = \tilde h_\gamma(k,l) - \tilde h_{\gamma'}(k',l') = \\
&\tilde h_\gamma(k,l) - \tilde h_{\gamma'}(k',l) + \tilde h_{\gamma'}(k',l) - \tilde h_{\gamma'}(k',l') = \\
&\tilde f(k,l) - \tilde f(k',l) + t_\gamma^{(0)}(k,l) - t_{\gamma'}^{(0)}(k',l) + \\
&\tilde h_{\gamma'}(k',l) - \tilde h_{\gamma'}(k',l') \ \text{ for } \ k-l = k'-l',
\end{aligned}
\tag{11.17}
$$

where

$$
\tilde h_\gamma(k,l) = h_\gamma(k,l) - \hat v(k-l), \quad \tilde f(k,l) = f(k,l) - \hat v(k-l), \tag{11.18}
$$

$k, l, k', l' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$.

Estimate (5.24) follows from (11.17), (11.18), (2.28b), (11.1b), (11.3), (2.31), (2.8), (5.1), (5.9a) and the formulas

$$
\begin{aligned}
&|k(\lambda,p,E) - k(\lambda',p,E)| = |l(\lambda,p,E) - l(\lambda',p,E)| = (E - p^2/4)^{1/2}|\lambda - \lambda'|, \\
&|\gamma^\pm(k(\lambda,p,E),p) - \gamma^\pm(k(\lambda',p,E),p)| \le |\lambda - \lambda'|,
\end{aligned}
\tag{11.19}
$$

$$
\Delta^s \ge \Delta^t \ \text{ for } \ 0 < s \le t, \ 0 \le \Delta \le 1, \ (E - p^2/4)^{1/2} \le E^{1/2}, \tag{11.20}
$$

where $k(\lambda,p,E)$ is given by (3.8), $l(\lambda,p,E) = k(\lambda,p,E) - p$, $\gamma^\pm(k,p)$ are given in (5.1), $\lambda, \lambda' \in \mathcal{T}$, $p \in \mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu$, $\tau \in]0,1[$. This completes the proof of (5.24).

To obtain (5.25b) we proceed from (5.11), (5.23), (5.24). First, using (5.23), (5.24) we obtain that: $H^0_{E,\tau}(\cdot,p)$ defined by (5.11a) is a bounded holomorphic function on $\mathcal{D}_+$ and is extended continuously on $\mathcal{D}_+ \cup \mathcal{T}$; $H^0_{E,\tau}(\cdot,p)$ defined by (5.11b) is a bounded holomorphic function on $\mathcal{D}_-$ and is extended continuously on $\mathcal{D}_- \cup \mathcal{T}$. Therefore, due to the maximum principle for holomorphic functions to prove (5.25b) it is sufficient to prove that

$$
|H^0_{E,\tau,\pm}(\lambda,p)| \le 2^{\mu/2}N + \frac{c_7(\alpha,\mu,\sigma,\beta)N^2}{(1-\eta)^2(1-\delta)E^{\beta/2}}, \quad \lambda \in \mathcal{T}, \quad p \in \mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu, \tag{11.21}
$$

where

$$
H^0_{E,\tau,\pm}(\lambda,p) = H^0_{E,\tau}(\lambda(1 \mp 0),p), \quad \lambda \in \mathcal{T}, \quad p \in \mathcal{B}_{2\tau\sqrt{E}} \backslash \mathcal{L}_\nu. \tag{11.22}
$$

Proceeding from (11.22), (5.11) we obtain that

$$
\begin{aligned}
&H^0_{E,\tau,+}(\lambda,p) = \hat v(p) + \frac{1}{2\pi i}\int\limits_{\mathcal{T}} (H_+(\zeta,p,E) - \hat v(p)) \frac{d\zeta}{\zeta - \lambda(1-0)} = \\
&\hat v(p) + \frac{1}{2}\big(H_+(\lambda,p,E) - \hat v(p)\big) + \frac{1}{2\pi i} p.v.\int\limits_{\mathcal{T}} (H_+(\zeta,p,E) - \hat v(p)) \frac{d\zeta}{\zeta - \lambda}, \\
&H^0_{E,\tau,-}(\lambda,p) = \hat v(p) - \frac{1}{2\pi i}\int\limits_{\mathcal{T}} (H_-(\zeta,p,E) - \hat v(p)) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda(1+0))} = \\
&\hat v(p) + \frac{1}{2}\big(H_-(\lambda,p,E) - \hat v(p)\big) - \frac{1}{2\pi i} p.v.\int\limits_{\mathcal{T}} (H_-(\zeta,p,E) - \hat v(p)) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)},
\end{aligned}
\tag{11.23}
$$

where $\lambda \in \mathcal{T}$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$.

Estimate (11.21) follows from formulas (11.23), where we use the decomposition

$$
\begin{aligned}
&p.v.\int_{\mathcal{T}} = p.v.\int_{\mathcal{T}_{\lambda,E}} + \int_{\mathcal{T}\backslash\mathcal{T}_{\lambda,E}},\\
&\mathcal{T}_{\lambda,E} = \{\zeta \in \mathcal{T}:\ |\zeta-\lambda| \le E^{-1/2}\},
\end{aligned}
\tag{11.24a}
$$

and from the estimation of $p.v. \int\limits_{\mathcal{T}_{\lambda,E}}$ using (5.23), (5.24) and the formulas

$$
\left|p.v. \int\limits_{\mathcal{T}_{\lambda,E}} \frac{d\zeta}{\zeta-\lambda}\right| \le \frac{c_{15}}{E^{1/2}},\quad \left|p.v. \int\limits_{\mathcal{T}_{\lambda,E}} \frac{d\zeta}{\zeta(\zeta-\lambda)}\right| \le \frac{c_{15}}{E^{1/2}},\quad E \ge 1, \tag{11.24b}
$$

$$
\int\limits_{\mathcal{T}_{\lambda,E}} \frac{|\zeta-\lambda|^\beta |d\zeta|}{|\zeta-\lambda|} \le \frac{c_{16}}{\beta E^{\beta/2}},\quad E \ge 1,\quad 0 < \beta \le 1, \tag{11.24c}
$$

and of $\int\limits_{\mathcal{T}\backslash\mathcal{T}_{\lambda,E}}$ using (5.23), (2.31) and the formula

$$
\int\limits_{\mathcal{T}\backslash\mathcal{T}_{\lambda,E}} \frac{|d\zeta|}{|\zeta-\lambda|} \le c_{17}\ln(1+E),\quad E \ge 1. \tag{11.24d}
$$

This completes the proof of (5.25b).

Lemma 5 is proved.

Proof of Lemma 9. Estimate (6.12) follows from (6.1), (6.4) and (2.39) for $\beta = 0$. Estimate (6.13) follows from (6.2), (6.12), (6.4), (2.39) for $\beta = \alpha$ (and the property $g_2 > g_1$).

Further, using (6.4) we obtain that

$$
h_\gamma(k,\cdot) - h_{\gamma'}(k',\cdot) = f(k,\cdot) - f(k',\cdot) + B_\gamma(k)h_\gamma(k,\cdot) - B_{\gamma'}(k')h_{\gamma'}(k',\cdot), \tag{11.25}
$$

$$
\begin{aligned}
&h_\gamma(k,\cdot) - h_{\gamma'}(k',\cdot) = f(k,\cdot) - f(k',\cdot) + (B_\gamma(k) - B_{\gamma'}(k'))h_{\gamma'}(k',\cdot) +\\
&B_\gamma(k)(h_\gamma(k,\cdot) - h_{\gamma'}(k',\cdot)),
\end{aligned}
\tag{11.26}
$$

where $k, k' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$. We consider (11.26) as a linear integral equation for $h_\gamma(k,\cdot) - h_{\gamma'}(k',\cdot)$. In addition:

$$
|f(k,l) - f(k',l)| \overset{(6.2)}{\le} \frac{g_2 N|k-k'|^\alpha}{(1+|k-l|^2)^{\mu/2}}, \tag{11.27}
$$

$$
|(B_\gamma(k) - B_{\gamma'}(k'))h_{\gamma'}(k',\cdot)(l)| \overset{(6.12),(11.8)}{\le} \frac{c_2'(\mu)c_{13}(\beta,\mu)(g_1N)^2|\gamma-\gamma'|^\beta}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}}, \tag{11.28}
$$

where $k, k', l \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $\gamma k = \gamma' k' = 0$, $|\gamma-\gamma'| \le 1$, $|k-k'| \le 1$, $0 < \beta \le 1/2$. Estimate (6.14) follows from (11.26)-(11.28) and (2.39) for $\beta = 0$.

Further, we have that

$$h_\gamma(k,l) - h_{\gamma'}(k',l') = (h_\gamma(k,l) - h_{\gamma'}(k',l)) + (h_{\gamma'}(k',l) - h_{\gamma'}(k',l')), \tag{11.29}$$

where $k, k', l, l' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$. Estimate (6.15) follows from (11.29), (6.13), (6.14), (2.8), (5.1), (5.9a) and (11.19).

To obtain (6.16) we proceed from (5.11), (2.8), (5.1), (5.9), (6.12), (6.15). In a similar way with the proof of (5.25b), to prove (6.16) it is sufficient to prove (6.16) for $H^0_{E,\tau}$ replaced by its limits

$$\begin{gathered}
H^0_{E,\tau,+}(\lambda,p) = H^0_{E,\tau}(\lambda(1-0),p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}} H_+(\zeta,p,E)\frac{d\zeta}{\zeta-\lambda(1-0)} = \\
\frac{1}{2}H_+(\lambda,p,E) + \frac{1}{2\pi i}p.v.\int\limits_{\mathcal{T}} H_+(\zeta,p,E)\frac{d\zeta}{\zeta-\lambda}, \\
H^0_{E,\tau,-}(\lambda,p) = H^0_{E,\tau}(\lambda(1+0),p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}} H_-(\zeta,p,E)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda(1+0))} = \\
\frac{1}{2}H_-(\lambda,p,E) + \frac{1}{2\pi i}p.v.\int\limits_{\mathcal{T}} H_-(\zeta,p,E)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)},
\end{gathered} \tag{11.30}$$

where $\lambda \in \mathcal{T}$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$. Estimate (6.16) (with $H^0_{E,\tau}$ replaced by $H^0_{E,\tau,\pm}$) follows from (11.30), where we use the decomposition (11.24), and from the estimation of $p.v. \int\limits_{\mathcal{T}_{\lambda,E}}$ using (6.12) (with (2.8), (5.1), (5.9)), (6.15), (11.24b), (11.24c) and of $\int\limits_{\mathcal{T}\backslash\mathcal{T}_{\lambda,E}}$ using (6.12) (with (2.8), (5.1), (5.9)), (11.24d).

Estimate (6.17) follows from (6.1), (6.7) and (2.39) for $\beta = 0$. Estimate (6.18) follows from (6.1), (6.7), (6.17) and (2.39) for $\beta = \alpha$.

Further, using (6.7) we obtain that

$$\begin{gathered}
t^{(n)}_\gamma(k,\cdot) - t^{(n)}_{\gamma'}(k',\cdot) = (B_\gamma(k))^{n+1}f(k,\cdot) - (B_{\gamma'}(k'))^{n+1}f(k',\cdot) + \\
B_\gamma(k)t^{(n)}_\gamma(k,\cdot) - B_{\gamma'}(k')t^{(n)}_{\gamma'}(k',\cdot),
\end{gathered} \tag{11.31}$$

$$\begin{gathered}
t^{(n)}_\gamma(k,\cdot) - t^{(n)}_{\gamma'}(k',\cdot) = ((B_\gamma(k))^{n+1} - (B_{\gamma'}(k'))^{n+1})f(k',\cdot) + \\
(B_\gamma(k))^{n+1}(f(k,\cdot) - f(k',\cdot)) + (B_\gamma(k) - B'_\gamma(k'))t^{(n)}_{\gamma'}(k',\cdot)) + \\
B_\gamma(k)(t^{(n)}_\gamma(k,\cdot) - t^{(n)}_{\gamma'}(k',\cdot)),
\end{gathered} \tag{11.32}$$

where $k, k' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $n \in \mathbb{N}\cup 0$. We consider (11.32) as a linear integral equation for $t^{(n)}_\gamma(k,\cdot) - t^{(n)}_{\gamma'}(k',\cdot)$. In addition, the following estimates hold:

$$|(B_\gamma(k))^{n+1}(f(k,\cdot) - f(k',\cdot))(l)| \le \frac{\delta_1^{n+1}g_2N|k-k'|^\alpha}{(1+|k-l|^2)^{\mu/2}}, \tag{11.33}$$

$$|(B_\gamma(k) - B_{\gamma'}(k'))t^{(n)}_{\gamma'}(k',\cdot)(l)| \le \frac{c_{13}(\beta,\mu)\delta_1^{n+1}(g_1N)^2|\gamma-\gamma'|^\beta}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}}, \tag{11.34}$$

$$|((B_\gamma(k))^{n+1} - (B_{\gamma'}(k'))^{n+1})f(k',\cdot)(l)| \le \frac{(n+1)c_{13}(\beta,\mu)\delta_1^n(g_1N)^2}{(1+|k-l|^2)^{\mu/2}}, \tag{11.35}$$

where $n \in \mathbb{N}\cup 0$, $k, k', l \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$, $\gamma k = \gamma' k' = 0$, $|\gamma - \gamma'| \le 1$, $|k-k'| \le 1$, $0 < \beta \le 1/2$. Estimate (11.33) follows from (6.2) and (2.39) for $\beta = 0$. Estimate (11.34) follows from (6.17), (11.8), (6.1).

To obtain (11.35) we use, in particular, the formulas

$$A_1^{n+1} - A_2^{n+1} = (A_1 - A_2)A_1^n + A_2(A_1^n - A_2^n), \tag{11.36}$$

$$\|A_1^{n+1} - A_2^{n+1}\| \le \|A_1 - A_2\|\|A_1\|^n + \|A_2\|\|A_1^n - A_2^n\|, \tag{11.37}$$

$$\|A_1^{n+1} - A_2^{n+1}\| \le (n+1)\|A_1 - A_2\|\,(\max(\|A_1\|, \|A_2\|))^n, \tag{11.38}$$

where A_1, A_2 are bounded linear operators in $C(\mathbb{S}^2_{\sqrt{E}})$, $\|\cdot\|$ denotes the norm of operators in $C(\mathbb{S}^2_{\sqrt{E}})$, $n \in \mathbb{N}\cup 0$. Note that (11.38) follows from (11.37) by the induction method. Estimate (11.35) follows from (6.1), (6.2), (11.8), (11.38) for $A_1 = \Lambda_k^\mu B_\gamma(k)\Lambda_k^{-\mu}$, $A_2 = \Lambda_k^\mu B_{\gamma'}(k')\Lambda_k^{-\mu}$, (2.38) and (2.39) for $\beta = 0$.

Estimate (6.19) follows from (11.32)-(11.35) and (2.39) for $\beta = 0$. Further, we have that

$$t^{(n)}_\gamma(k,l) - t^{(n)}_{\gamma'}(k',l') = (t^{(n)}_\gamma(k,l) - t^{(n)}_{\gamma'}(k',l)) + (t^{(n)}_{\gamma'}(k',l) - t^{(n)}_{\gamma'}(k',l')), \tag{11.39}$$

where $n \in \mathbb{N}\cup 0$, $k, k', l, l' \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma, \gamma' \in \mathbb{S}^2$. Estimate (6.20) follows from (11.39), (6.18), (6.19), (6.9), (11.19), (2.28c), (2.28d).

To obtain (6.21) we proceed from (6.9), (6.17), (6.20) and the definition of $T^{0,n}_{E,\tau}$. In a similar way with the proofs of (5.25b) and (6.16), to prove (6.21) it is sufficient to prove (6.21) for $T^{0,n}_{E,\tau}$ replaced by its limits

$$\begin{aligned} T^{0,n}_{E,\tau,+}(\lambda,p) &= T^{0,n}_{E,\tau}(\lambda(1-0),p) = \frac{1}{2\pi i}\int_{\mathcal{T}} T^{(n)}_+(k(\zeta,p,E),p)\frac{d\zeta}{\zeta - \lambda(1-0)}, \\ T^{0,n}_{E,\tau,-}(\lambda,p) &= T^{0,n}_{E,\tau}(\lambda(1+0),p) = \frac{1}{2\pi i}\int_{\mathcal{T}} T^{(n)}_-(k(\zeta,p,E),p)\frac{\lambda d\zeta}{\zeta(\zeta - \lambda(1+0))}, \end{aligned} \tag{11.40}$$

where $\lambda \in \mathcal{T}$, $p \in \mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu$. Further, the proof of (6.21) (with $T^{0,n}_{E,\tau}$ replaced by $T^{0,n}_{E,\tau,\pm}$) is completely similar to the proof of (6.16) (with $H^0_{E,\tau}$ replaced by $H^0_{E,\tau,\pm}$).

Lemma 9 is proved.

Proof of Lemma 10. Using (6.4) we obtain that

$$h_\gamma(k,\cdot) - \tilde h_\gamma(k,\cdot) = f(k,\cdot) - \tilde f(k,\cdot) + B_\gamma(k)h_\gamma(k,\cdot) - \tilde B_\gamma(k)\tilde h_\gamma(k,\cdot), \tag{11.41}$$

$$\begin{aligned} &h_\gamma(k,\cdot) - \tilde h_\gamma(k,\cdot) = f(k,\cdot) - \tilde f(k,\cdot) + (B_\gamma(k) - \tilde B_\gamma(k))\tilde h_\gamma(k,\cdot) + \\ &B_\gamma(k)(h_\gamma(k,\cdot) - \tilde h_\gamma(k,\cdot)), \end{aligned} \tag{11.42}$$

where $k \in \mathbb{S}^2_{\sqrt{E}}$, $\gamma \in \mathbb{S}^2$, $\tilde B_\gamma(k)$ is defined by (2.38) with f replaced by $\tilde f$. We consider (11.42) as a linear integral equation for $h_\gamma(k,\cdot) - \tilde h_\gamma(k,\cdot)$. Using estimate (6.12) with $h_\gamma(k,l)$ replaced by $\tilde h_\gamma(k,l)$ and estimate (2.39) (for $\beta = 0$) with f replaced by $f - \tilde f$, we obtain that

$$\begin{aligned}
&|(B_\gamma(k) - \tilde B_\gamma(k))\tilde h_\gamma(k,\cdot)(l)| \le \frac{c_3(0,\mu,\sigma,3)g_1 N\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}}{E^{\sigma/2}(1-\delta_1)(1+|k-l|^2)^{\mu/2}}\\
&\overset{(6.11)}{\le} \frac{\delta_1\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}}{(1-\delta_1)(1+|k-l|^2)^{\mu/2}},
\end{aligned}\tag{11.43}$$

where $\gamma \in \mathbb{S}^2$, $k, l \in \mathbb{S}^2_{\sqrt{E}}$. Estimate (6.22) follows from (11.42), (11.43) and (2.39) for $\beta = 0$.

To prove (6.23) we proceed from (6.15), (6.22) and the definitions of $H_\pm$ and $\tilde H_\pm$. Due to (6.22) and the definitions of $H_\pm$ and $\tilde H_\pm$, we have that

$$\begin{aligned}
&|(H_\pm - \tilde H_\pm)(k(\lambda,p,E),p) - (H_\pm - \tilde H_\pm)(k(\lambda',p,E),p)| \le\\
&\frac{2\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}}{(1-\delta_1)^2(1+p^2)^{\mu/2}},
\end{aligned}\tag{11.44}$$

where $\lambda, \lambda' \in \mathcal{T}$, $p \in \mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$. Using (6.15) we obtain that

$$\begin{aligned}
&|(H_\pm - \tilde H_\pm)(k(\lambda,p,E),p) - (H_\pm - \tilde H_\pm)(k(\lambda',p,E),p)| =\\
&|H_\pm(k(\lambda,p,E),p) - H_\pm(k(\lambda',p,E),p) - (\tilde H_\pm(k(\lambda,p,E),p) - \tilde H_\pm(k(\lambda',p,E),p))|\\
&\le 2\left(\frac{c_{10}(\mu)(1+\delta_2)g_2N}{1-\delta_1} + \frac{c_9(\beta,\mu)(g_1N)^2}{(1-\delta_1)^2E^{\beta/2}}\right)\frac{E^{\beta/2}|\lambda-\lambda'|^\beta}{(1+p^2)^{\mu/2}},
\end{aligned}\tag{11.45}$$

where $\lambda, \lambda' \in \mathcal{T}$, $p \in \mathcal{B}_{2\sqrt{E}}\backslash\mathcal{L}_\nu$, $|\lambda - \lambda'| \le (E - p^2/4)^{-1/2}$, $0 < \beta < \min(\alpha, 1/2)$. In addition:

$$\begin{aligned}
&\|f-\tilde f\|_{C(\mathcal{M}_E),\mu} = \big(\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}\big)^{1-\varepsilon}\big(\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}\big)^{\varepsilon} \le\\
&\big(\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}\big)^{1-\varepsilon}\big(E^{\beta/2}|\lambda-\lambda'|^\beta\big)^{\varepsilon} \ \text{ for } \ \|f-\tilde f\|_{C(\mathcal{M}_E),\mu} \le E^{\beta/2}|\lambda-\lambda'|^\beta,
\end{aligned}\tag{11.46}$$

$$\begin{aligned}
&E^{\beta/2}|\lambda-\lambda'|^\beta = \big(E^{\beta/2}|\lambda-\lambda'|^\beta\big)^{1-\varepsilon}\big(E^{\beta/2}|\lambda-\lambda'|^\beta\big)^{\varepsilon} \le\\
&\big(\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}\big)^{1-\varepsilon}\big(E^{\beta/2}|\lambda-\lambda'|^\beta\big)^{\varepsilon} \ \text{ for } \ \|f-\tilde f\|_{C(\mathcal{M}_E),\mu} \ge E^{\beta/2}|\lambda-\lambda'|^\beta,
\end{aligned}\tag{11.47}$$

where $0 \le \varepsilon \le 1$. Estimate (6.23) follows from (11.44)-(11.47).

To obtain (6.24) we proceed from (6.22), (6.23) and the definitions of $H_\pm$, $\tilde H_\pm$, $H^0_{E,\tau}$, $\tilde H^0_{E,\tau}$. This proof is completely similar to the proof of (6.16).

Lemma 10 is proved.

Proof of Lemma 11. Estimate (6.27) follows from (2.36) (for $\alpha = 0$), (6.1) and the property

$$|u(|k-l|, 2\tau_0\sqrt{E}, 2\tau\sqrt{E})| \le 1, \quad (k,l) \in \mathcal{M}_E. \tag{11.48}$$

Estimate (6.28) follows from (2.36), (6.1), (6.2), (6.27), the formula

$$\tilde f(k,l)-\tilde f(k',l')=(u(|k-l|,2\tau_0\sqrt{E},2\tau\sqrt{E})-u(|k'-l'|,2\tau_0\sqrt{E},2\tau\sqrt{E}))f(k,l)+\\ u(|k'-l'|,2\tau_0\sqrt{E},2\tau\sqrt{E})(f(k,l)-f(k',l')),\ \ (k,l)\in\mathcal{M}_E,\ \ (k',l')\in\mathcal{M}_E, \tag{11.49}$$

property (11.48) and the inequalities

$$|u(|k-l|,2\tau_0\sqrt{E},2\tau\sqrt{E})-u(|k'-l'|,2\tau_0\sqrt{E},2\tau\sqrt{E})|\le\\ \frac{|\,|k-l|-|k'-l'|\,|}{2(\tau-\tau_0)\sqrt{E}}\le\frac{|k-k'|+|l-l'|}{2(\tau-\tau_0)\sqrt{E}},\ \ (k,l)\in\mathcal{M}_E,\ \ (k',l')\in\mathcal{M}_E. \tag{11.50}$$

Estimate (6.29) follows from (2.36) (for $\alpha=0$), (6.1) and the formulas

$$(1+|k-l|^2)^{\mu_0/2}|f(k,l)-\tilde f(k,l)|\le\frac{\|f\|_{C(\mathcal{M}_E),\mu}(1-u(|k-l|,2\tau_0\sqrt{E},2\tau\sqrt{E}))}{(1+|k-l|^2)^{(\mu-\mu_0)/2}}, \tag{11.51}$$

$$1-u(|k-l|,2\tau_0\sqrt{E},2\tau\sqrt{E})=0\ \text{ for }\ |k-l|\le 2\tau_0\sqrt{E},\\ |1-u(|k-l|,2\tau_0\sqrt{E},2\tau\sqrt{E})|\le 1, \tag{11.52}$$

where $(k,l)\in\mathcal{M}_E$.

Lemma 11 is proved.

12. Proof of Lemmas 7 and 8

Proof of Lemma 7. For

$$U^0,U,U_1,U_2\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)),\\ |||U^0|||_{E,\tau,\mu}\le r/2,\ \ |||U|||_{E,\tau,\mu}\le r,\ \ |||U_1|||_{E,\tau,\mu}\le r,\ \ |||U_2|||_{E,\tau,\mu}\le r, \tag{12.1}$$

using Lemma 4 and the assumptions of Lemma 7 we obtain that

$$M_{E,\tau,U^0}(U)\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu)),\\ |||M_{E,\tau,U^0}(U)|||_{E,\tau,\mu}\le|||U^0|||_{E,\tau,\mu}+|||M_{E,\tau}(U)|||_{E,\tau,\mu}\le\\ r/2+c_5c_4(\mu,\tau,E)r^2<r, \tag{12.2}$$

$$|||M_{E,\tau,U^0}(U_1)-M_{E,\tau,U^0}(U_2)|||_{E,\tau,\mu}\le 2c_5c_4(\mu,\tau,E)r\,|||U_1-U_2|||_{E,\tau,\mu},\\ 2c_5c_4(\mu,\tau,E)r<1, \tag{12.3}$$

where

$$M_{E,\tau,U^0}(U)=U^0+M_{E,\tau}(U). \tag{12.4}$$

Due to (12.1)-(12.4), M_{E,τ,U^0} is a contraction map of the ball $U\in L^\infty((\mathbb{C}\backslash(\mathcal{T}\cup 0))\times(\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu))$, $|||U|||_{E,\tau,\mu}\le r$. Using now the lemma about contraction maps and using the formulas

$$|||U-M^n_{E,\tau,U^0}(0)|||_{E,\tau,\mu}\le\sum_{j=n}^{\infty}|||M^{j+1}_{E,\tau,U^0}(0)-M^{j}_{E,\tau,U^0}(0)|||_{E,\tau,\mu}, \tag{12.5}$$

$$|||M_{E,\tau,U^0}(0)-0|||_{E,\tau,\mu}=|||U^0|||_{E,\tau,\mu}\le r/2, \tag{12.6}$$

$$|||M^{j+1}_{E,\tau,U^0}(0) - M^{j}_{E,\tau,U^0}(0)|||_{E,\tau,\mu} \overset{(12.3)}{\leq} 2c_5c_4(E,\tau,\mu)r\times \\ |||M^{j}_{E,\tau,U^0}(0) - M^{j-1}_{E,\tau,U^0}(0)|||_{E,\tau,\mu}, \quad j=1,2,3..., \tag{12.7}$$

where $M^0_{E,\tau,U^0}(0) = 0$, we obtain Lemma 7.

Proof of Lemma 8. We have that

$$U - \tilde{U} = U^0 - \tilde{U}^0 + M_{E,\tau}(U) - M_{E,\tau}(\tilde{U}), \tag{12.8a}$$

$$M_{E,\tau}(U) - M_{E,\tau}(\tilde{U}) = M_{E,\tau}(U-\tilde{U},U) + M_{E,\tau}(\tilde{U},U-\tilde{U}), \tag{12.8b}$$

where

$$M_{E,\tau}(U_1,U_2)(\lambda,p) = M^+_{E,\tau}(U_1,U_2)(\lambda,p) = \\ -\frac{1}{\pi}\iint\limits_{\mathcal{D}_+}(U_1,U_2)_{E,\tau}(\zeta,p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta-\lambda}, \quad \lambda\in\mathcal{D}_+\backslash 0, \quad p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{12.9a}$$

$$M_{E,\tau}(U_1,U_2)(\lambda,p) = M^-_{E,\tau}(U_1,U_2)(\lambda,p) = \\ -\frac{1}{\pi}\iint\limits_{\mathcal{D}_-}(U_1,U_2)_{E,\tau}(\zeta,p)\frac{\lambda dRe\,\zeta dIm\,\zeta}{\zeta(\zeta-\lambda)}, \quad \lambda\in\mathcal{D}_-, \quad p\in\mathcal{B}_{2\tau\sqrt{E}}\backslash\mathcal{L}_\nu, \tag{12.9b}$$

where $(U_1,U_2)(\zeta,p)$ is defined by (5.13).

In view of (12.8b) we can consider (12.8a) as a linear integral equation for "unknown" $U-\tilde{U}$ with given $U^0-\tilde{U}^0$, U, $\tilde{U}$.

As well as (5.19), using also Lemma 7 we obtain that

$$|||M_{E,\tau}(U-\tilde{U},U) - M_{E,\tau}(\tilde{U},U-\tilde{U})|||_{E,\tau,\mu} \leq 2c_5c_4(\mu,\tau,E)r|||U-\tilde{U}|||_{E,\tau,\mu}. \tag{12.10}$$

Using (12.8b), (12.10) and solving (12.8a) by the method of successive approximations we obtain (5.29). Lemma 8 is proved.

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